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Higher-order Boussinesq-type equations for surface gravity waves: derivation and analysis

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Contents

Boussinesq-type equations of higher order in dispersion as well as in nonlinearity are derived for waves (and wave–current interaction) over an uneven bottom. Formulations are given in terms of various velocity variables such as the depth-averaged velocity and the particle velocity at the still water level, and at an arbitrary vertical location. The equations are enhanced and analysed with emphasis on linear dispersion, shoaling and nonlinear properties for large wavenumbers.

As a starting point the velocity potential is expanded as a power series in the vertical coordinate measured from the still water level (SWL). Substituting this expansion into the Laplace equation leads to a velocity field expressed in terms of spatial derivatives of the vertical velocity $\hat{w}$ and the horizontal velocity vector $\hat{\boldsymbol{u}}$ at the SWL. The series expressions are given to infinite order in the dispersion parameter, μ. Satisfying the kinematic bottom boundary condition defines an implicit relation between $\hat{w}$ and $\hat{\boldsymbol{u}}$, which is recast as an explicit recursive expression for $\hat{w}$ in terms of $\hat{\boldsymbol{u}}$ under the assumption that $\mu \ll 1$. Boussinesq equations are then derived from the dynamic and kinematic boundary conditions at the free surface. In this process the infinite series solutions are truncated at $O(\mu^6)$, while all orders of the nonlinearity parameter, ϵ are included to that order in dispersion. This leads to a set of higher-order Boussinesq equations in terms of the surface elevation η and the horizontal velocity vector $\hat{\boldsymbol{u}}$ at the SWL.

The equations are recast in terms of the depth-averaged velocity, $\boldsymbol{U}$ leaving out $O(\epsilon^2\mu^4)$, which corresponds to assuming $\epsilon = O(\mu)$. This formulation turns out to include singularities in linear dispersion as well as in nonlinearity. Next, the technique introduced by Madsen *et al.* in 1991 and Schäffer & Madsen in 1995 is invoked, and this results in a set of enhanced equations formulated in $\boldsymbol{U}$ and including $O(\mu^4, \epsilon\mu^4)$ terms. These equations contain no singularities and the embedded linear and nonlinear properties are shown to be significantly improved. To quantify the accuracy, Stokes's third-order theory is used as reference and Fourier analyses of the new equations are carried out to third order (in nonlinearity) for regular waves on a constant depth and to first order for shoaling characteristics. Furthermore, analyses are carried out to second order for bichromatic waves and to first order for waves in ambient currents. These analyses are not restricted to small values of the linear dispersion parameter, μ. In conclusion, the new equations are shown to have linear dispersion characteristics corresponding to a Padé [4,4] expansion in $k'h'$ (wavenumber times depth) of the squared celerity according to Stokes's linear theory. This corresponds to a quite high accuracy in linear dispersion up to approximately $k'h' = 6$. The high quality of dispersion is also achieved for the Doppler shift in connection with wave–current interaction and it allows for a study of wave blocking due to opposing currents. Also, the linear shoaling characteristics are shown to be excellent, and the accuracy of nonlinear transfer of energy to sub- and superharmonics is found to be superior to previous formulations.

The equations are then recast in terms of the particle velocity, $\tilde{\boldsymbol{u}}$, at an arbitrary vertical location including $O(\mu^4, \epsilon^5\mu^4)$ terms. This formulation includes, as special subsets, Boussinesq equations in terms of the bottom velocity or the surface velocity. Furthermore, the arbitrary location of the velocity variable can be used to optimize the embedded linear and nonlinear characteristics. A Fourier analysis is again carried out to third order (in ϵ) for regular waves. It turns out that Padé [4,4] linear dispersion characteristics can not be achieved for any choice of the location of the velocity variable. However, for an optimized location we achieve fairly good linear characteristics and very good nonlinear characteristics.

Finally, the formulation in terms of $\tilde{\boldsymbol{u}}$ is modified by introducing the technique of dispersion enhancement while retaining only $O(\mu^2, \epsilon^3\mu^2)$ terms. Now the resulting set of equations do show Padé [4,4] dispersion characteristics in the case of pure waves as well as in connection with ambient currents, and again the nonlinear properties (such as second- and third-order transfer functions and amplitude dispersion) are shown to be superior to those of existing formulations of Boussinesq-type equations.

Keywords: Boussinesq equations; surface gravity waves; nonlinear random waves; wave–current interactions; triad interactions; nonlinear dispersive waves

1. Introduction

The irrotational motion of an incompressible homogeneous inviscid fluid is generally a three-dimensional problem. The main issue of Boussinesq-type equations is, however, to reduce the description to a two-dimensional one by introducing a polynomial approximation of the vertical distribution of the flow field into the integral conservation laws, while accounting for non-hydrostatic effects due to the vertical acceleration of water.

There are two important parameters associated with Boussinesq-type equations: the nonlinearity, represented by the ratio of amplitude to depth, ϵ; and the dispersion, represented by the ratio of depth to wave length, μ. The derivation generally requires an assumption of $\mu \ll 1$, while ϵ may be taken as arbitrary (see, for example, Mei 1983). However, in the classical Boussinesq equations it is assumed that ϵ and μ^2 are of the same order. This assumption represents a balance between lowest-order dispersion and lowest-order nonlinearity and allows for wave solutions of constant form (e.g. cnoidal waves).

In the original work by Boussinesq (1872), which was restricted to a horizontal bottom, the vertical velocity was simply assumed to vary linearly from zero at the bottom to a maximum at the free surface, and the non-hydrostatic pressure was a consequence of the local acceleration of this velocity. In 1966, Mei & LeMéhauté extended this work to an uneven bottom in one dimension (with later corrections given by Madsen & Mei (1969)) and in 1967, Peregrine derived what are now referred to as the 'classical' Boussinesq equations in two horizontal dimensions for the case of an uneven bottom. Peregrine presented two versions of the equations, one given in terms of the velocity vector at the still water level, and one in terms of the depth-averaged velocity vector.

There is in fact no unique form of the classical Boussinesq equations. First of all the choice of velocity variable is arbitrary and secondly, since the nonlinear and dispersive terms are of higher order than the leading terms, they can be manipulated by invoking the linear long-wave equation (see, for example, Peregrine 1967; Whitham 1974; Svendsen 1974; Mei 1983). It has been demonstrated by, for example, Mei (1983) and Madsen *et al*. (1991), that the accuracy of the linear dispersion characteristics for larger wavenumbers is sensitive to the choice of velocity variable and to the mixture of spatial and temporal derivatives in the governing equations.

For many practical applications of Boussinesq equations, the weak dispersion is the most critical limitation, and this problem has achieved considerable attention in

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the past five to ten years. Several alternative equations have been presented with the purpose of improving the dispersion characteristics (see, for example, Witting 1984; Madsen *et al*. 1991; Madsen & Sørensen 1992; Nwogu 1993; Schröter *et al*. 1994; Schäffer & Madsen 1995*a*).

Witting (1984) was the first to demonstrate the efficiency of Padé approximations in connection with the linear celerity, and Madsen *et al*. (1991) used this idea to derive a new set of lower-order equations with dispersion characteristics corresponding to Padé [2,2] expansion in $k'h'$ of the squared celerity according to Stokes's linear theory. The technique may be viewed as an interpolation between two different formulations of the dispersive terms. This is equivalent to manipulating the dispersive terms using the linear long-wave equations. In practice, the approach by Madsen *et al*. (1991) was to truncate the momentum equation at $O(\mu^2)$, to apply spatial differentiation twice, to multiply with a free coefficient and finally to add the result of order $O(\mu^4)$ to the original momentum equation. This procedure resulted in a mixture of spatial and temporal third derivatives, and the free coefficient was fixed to match the Padé approximation. The same dispersion relation was achieved by Nwogu (1993) using a different approach: he introduced a free parameter by choosing the velocity variable at an arbitrary z location, and the resulting equations contained third derivative terms in the mass as well as in the momentum equation. Again the free coefficient was chosen to match the Padé [2,2] approximation. Recently, Schäffer & Madsen (1995*a*) combined these two approaches to derive a set of lower-order Boussinesq equations with linear dispersion characteristics corresponding to a Padé [4,4] expansion (see also Schröter *et al*. 1994). This technique will be pursued further in the present paper.

Linear dispersion in connection with wave–current interaction has achieved much less attention, although it is well known that one consequence of the nonlinearity of the Boussinesq equations is the automatic inclusion of wave-averaged effects such as radiation stress. This is not, however, a guarantee for a correct representation of, for example, the Doppler shift in connection with current refraction, and in fact most Boussinesq-type equations fail to model this phenomenon accurately. Yoon & Liu (1989) were the first to address the problem and found that additional terms had to be included in the classical equations to achieve a Doppler shift correct to lowest order in dispersion. However, in opposing currents this dispersion accuracy is still far from satisfactory, as the wavenumbers increase rapidly with the Froude number and consequently violate the validity of the equations. Recently, the equations by Yoon & Liu have been extended by Kristensen (1995) to achieve a Doppler shift of the correct form and with a dispersion relation corresponding to a Padé [2,2] expansion. In this paper we shall match a Padé [4,4] expansion as also pursued by Chen *et al*. (1998) focusing directly on the wave–current problem.

The nonlinear properties of Boussinesq-type equations have been analysed by, for example, Madsen & Sørensen (1993), who presented evolution equations for triad interactions and second-order transfer functions for sub- and superharmonics. On the basis of a depth-integrated velocity formulation of the Boussinesq equations they found that for increasing wavenumbers ($k'h'$) the bound second harmonic tends to be clearly underestimated compared to Stokes's second-order theory. It turns out, however, that the nonlinear properties are also sensitive to the choice of velocity variable, and using, for example, the depth-averaged velocity, leads to an overestimation of the second harmonic for $k'h'$ values in the interval 0.25–1.5. Improvement of the nonlinear properties basically requires the inclusion of terms which combine the effects of nonlinearity and dispersion (e.g. $\epsilon\mu^2$, $\epsilon^2\mu^2$ and $\epsilon\mu^4$ terms). Such formulations will be pursued in the present paper.

The various Boussinesq-type equations discussed so far have all been based on the assumption that $\epsilon = O(\mu^2)$, although this assumption is not strictly neccessary in the derivation. In 1953, Serre presented an alternative to the original Boussinesq theory by combining lowest-order dispersion with full nonlinearity: as a basic assumption Serre used a horizontal velocity, which was independent of z. He then determined the vertical velocity by integration of the local continuity equation and the pressure from integration of the vertical momentum equation. In contrast to Boussinesq, Serre retained all nonlinear terms such as the convective vertical acceleration terms. For a number of years the work by Serre achieved little attention outside France, which is illustrated by the fact that almost identical equations valid on a horizontal bottom were presented in 1969 by Su & Gardner. A critical review and a rederivation of Serre's equations can be found in Dingemans (1994). Recently, Wei *et al*. (1995) presented a set of so-called 'fully nonlinear' Boussinesq equations. These equations are derived on the basis of the classical expansion of the horizontal velocity profile rather than the simple uniform velocity used by Serre. The order of ϵ is taken to be arbitrary and all nonlinear terms at order $O(\mu^2)$ are included. Except for being formulated in terms of the velocity vector at an arbitrary z level (giving an important improvement of the linear dispersion characteristics), the equations of Wei *et al*. are basically equivalent to the equations introduced by Serre (1953).

Boussinesq equations of higher order in dispersion as well as in nonlinearity are rare, and we only know of the pioneering work by Dingemans (1973), who in one dimension retained terms of order $O(\mu^4)$ and $O(\epsilon\mu^2)$ assuming that $\epsilon = O(\mu^2)$. The equations were given in two versions, one based on the depth-averaged velocity and one based on the velocity at the still water level. Dingemans did not provide analyses or computations based on these equations. In § 3 we shall demonstrate that the equations expressed in terms of the depth-averaged velocity are unstable due to singularities in the embedded dispersion relation. This makes these equations useless. The equations expressed in terms of the velocity at the still water level are analysed in § 5.

To summarize, the classical Boussinesq equations include terms of order $O(\mu^2)$ and $O(\epsilon)$, and assume that $\epsilon = O(\mu^2)$. In the higher-order (one-dimensional) Boussinesq equations of Dingemans (1973), it is still assumed that $\epsilon = O(\mu^2)$ and terms of order $O(\mu^4)$ and $O(\epsilon\mu^2)$ are retained. The equations of Serre (1953) and Wei *et al*. (1995) retain terms of order $O(\mu^2)$, while ϵ is arbitrary. In line with recent publications, we prefer to denote all these different forms as 'Boussinesq-type' equations, as they are basically derived from the same starting point: a polynomial expansion in the vertical coordinate.

Other evolution equations for nonlinear waves have been presented by Kennedy & Fenton (1995) and Isobe (1994). Both approaches rely on solutions to the Laplace equation which are polynomials in the vertical coordinate and as such they have a strong resemblance to Boussinesq-type derivations.

The present paper is organized as follows. In § 2 we define the governing equations and scaling parameters, expand the velocity potential in power series, and derive a set of higher-order equations in terms of the velocity vector at the still water level. In §§ 3 and 4 we reformulate the equations in terms of the depth-averaged velocity vector. The straightforward formulation is found to have rather poor characteristics and a pernicious singularity (§ 3), and therefore new enhanced equations are derived and analysed in § 4. In §§ 5 and 6 we reformulate the equations in terms of the velocity vector at an arbitrary z location. The straightforward higher-order formulation is derived and analysed in § 5, while an enhanced set of lower-order equations is derived and analysed in § 6. The analyses performed in §§ 3, 4, 5 and 6 include linear dispersion and shoaling characteristics, and second- and third-order transfer functions for regular waves. In § 7 the analysis of the various Boussinesq-type equations is extended to second-order transfer for sub- and superharmonics for bichromatic primary waves. Finally, § 8 concentrates on the interaction between waves and an ambient current, which is uniform over depth and slowly varying in time and space. Summary and conclusions are given in § 9.

2. Equations in terms of the velocity at the still water level, $\hat{\boldsymbol{u}}$

(a) Scaling and governing equations

We adopt a Cartesian coordinate system with the x'-axis and y'-axis located on the still water plane (SWL) and with the z'-axis pointing vertically upwards. The fluid domain is bounded by the sea bed at $z' = -h'(x', y')$ and the free surface $z' = \eta'(x', y', t')$. Non-dimensional variables are used as follows:

$$x = \frac{x'}{l'_0}, \quad y = \frac{y'}{l'_0}, \quad z = \frac{z'}{h'_0}, \quad t = \frac{\sqrt{gh'_0}}{l'_0}t', \tag{2.1 a}$$

$$h = \frac{h'}{h'_0}, \quad \eta = \frac{\eta'}{a'_0}, \quad \Phi = \frac{h'_0}{a'_0 l'_0 \sqrt{gh'_0}}\Phi', \tag{2.1 b}$$

where prime denotes dimensional variables and h'_0, l'_0 and a'_0 denote a characteristic water depth, wavelength and wave amplitude, respectively. The velocity potential Φ is related to the velocity components by the definition

$$\boldsymbol{u} \equiv \nabla\Phi, \quad w \equiv \Phi_z, \tag{2.2}$$

where ∇ is the two-dimensional gradient operator, which in Cartesian coordinates reads $(\partial/\partial_x, \partial/\partial_y)$. As a result of the scaling the governing equations and boundary conditions for the irrotational wave problem read

$$\Phi_{zz} + \mu^2\nabla^2\Phi = 0, \quad -h < z < \epsilon\eta, \tag{2.3 a}$$

$$(1/\mu^2)\Phi_z + \nabla h \cdot \nabla\Phi = 0, \quad z = -h, \tag{2.3 b}$$

$$\Phi_t + \eta + \tfrac{1}{2}\epsilon((\nabla\Phi)^2 + (1/\mu^2)(\Phi_z)^2) = 0, \quad z = \epsilon\eta, \tag{2.3 c}$$

$$-(1/\mu^2)\Phi_z + \eta_t + \epsilon\nabla\eta \cdot \nabla\Phi = 0, \quad z = \epsilon\eta, \tag{2.3 d}$$

where ϵ and μ are the classical measures of nonlinearity and frequency dispersion defined by

$$\epsilon = \frac{a'_0}{h'_0}, \quad \mu = \frac{h'_0}{l'_0}. \tag{2.4}$$

(*b*) *Power series solution to the Laplace equation*

As discussed in the introduction, one of the main ideas in Boussinesq-type derivations is to reduce the three-dimensional description to a two-dimensional one. The first step towards such a reduction is to apply separation of variables and to introduce an expansion of the velocity potential as a power series in the vertical coordinate. On a horizontal bottom it is convenient to expand in powers of $(z + h)$ rather than in z, but in the general case this advantage vanishes. We choose the expansion

$$\Phi(x, y, z, t) = \sum_{n=0}^{\infty} z^n \Phi^{(n)}(x, y, t), \tag{2.5}$$

where the orders of magnitude in μ of $\Phi^{(n)}$ are yet to be determined. Chen & Liu (1995) used an alternative approach in which the velocity potential was expanded in powers of μ and the z variation was determined by integration. Eventually, however, the two approaches lead to identical results. Using (2.5) in (2.3 *a*) leads to a polynomial in z and by requiring the coefficient of each power of z to vanish, we obtain the classical recurrence relation

$$\Phi^{(n+2)} = -\mu^2 \frac{\nabla^2 \Phi^{(n)}}{(n+1)(n+2)}, \quad n = 1, 2, 3, \ldots . \tag{2.6}$$

In combination with (2.5) this leads to the general result

$$\Phi(x, y, z, t) = \sum_{n=0}^{\infty} (-1)^n \mu^{2n} \left(\frac{z^{2n}}{(2n)!} \nabla^{2n} \Phi^{(0)} + \frac{z^{2n+1}}{(2n+1)!} \nabla^{2n} \Phi^{(1)} \right), \tag{2.7}$$

which is a series solution with only two unknown functions $\Phi^{(0)}$ and $\Phi^{(1)}$. By the use of (2.7) and (2.2) the velocity field can be expressed by

$$\boldsymbol{u}(x, y, z, t) = \sum_{n=0}^{\infty} (-1)^n \left(\frac{z^{2n}}{(2n)!} \mu^{2n} \nabla (\nabla^{2n-2} (\nabla \cdot \hat{\boldsymbol{u}})) + \frac{z^{2n+1}}{(2n+1)!} \mu^{2n+2} \nabla (\nabla^{2n} \hat{w}) \right), \tag{2.8 a}$$

$$w(x, y, z, t) = \sum_{n=0}^{\infty} (-1)^n \left(-\frac{z^{2n+1}}{(2n+1)!} \mu^{2n+2} \nabla^{2n} (\nabla \cdot \hat{\boldsymbol{u}}) + \frac{z^{2n}}{(2n)!} \mu^{2n+2} \nabla^{2n} \hat{w} \right), \tag{2.8 b}$$

where

$$\hat{\boldsymbol{u}} \equiv \nabla \Phi^{(0)}, \quad \hat{w} \equiv (1/\mu^2) \Phi^{(1)}. \tag{2.9}$$

The physical interpretation of $\hat{\boldsymbol{u}}$ and $\hat{w}$ is obtained by setting $z = 0$ in (2.8) which yields

$$\boldsymbol{u}(x, y, 0, t) \equiv \hat{\boldsymbol{u}}, \quad w(x, y, 0, t) \equiv \mu^2 \hat{w}. \tag{2.10}$$

Hence (2.7) and (2.8) define the wave kinematics in terms of the velocity components at the still water datum. These kinematics satisfy the Laplace equation, but so far no boundary conditions have been involved.

(c) The kinematic bottom boundary condition

The relation between the horizontal and vertical velocity components at the still water level can be established by applying the kinematic boundary condition at the sea bed. Inserting (2.7) and (2.9) in (2.3 b) yields

$$\sum_{n=0}^{\infty}(-1)^n\left(\mu^{2n}\frac{h^{2n+1}}{(2n+1)!}\nabla^{2n}(\nabla\cdot\hat{\boldsymbol{u}})+\mu^{2n}\frac{h^{2n}}{(2n)!}\nabla^{2n}\hat{w}\right)$$
$$+\nabla h\cdot\sum_{n=0}^{\infty}(-1)^n\left(\mu^{2n}\frac{h^{2n}}{(2n)!}\nabla(\nabla^{2n-2}(\nabla\cdot\hat{\boldsymbol{u}}))-\mu^{2n+2}\frac{h^{2n+1}}{(2n+1)!}\nabla(\nabla^{2n}\hat{w})\right)=0, \tag{2.11 a}$$

which may be written in the alternative form

$$\hat{w}+\sum_{n=0}^{\infty}(-1)^n\mu^{2n}\nabla\cdot\left(\frac{h^{2n+1}}{(2n+1)!}(\nabla(\nabla^{2n-2}(\nabla\cdot\hat{\boldsymbol{u}})))-\mu^2\frac{h^{2n+2}}{(2n+2)!}\nabla(\nabla^{2n}\hat{w})\right)=0. \tag{2.11 b}$$

So far no assumptions concerning the expansion parameters μ and ϵ have been introduced and in that sense all expressions are still exact. However, in order to establish an expression for $\hat{w}$ in terms of $\hat{\boldsymbol{u}}$ we introduce the expansion

$$\hat{w}(x,y,t)=\sum_{m=0}^{\infty}\mu^{2m}w^{(m)}(x,y,t) \tag{2.12}$$

and assume that $\mu \ll 1$. We insert (2.12) in (2.11 b), which leads to two single summation series and one double summation series. In the single summations, m and n are replaced by p, while m is replaced by $p-n-1$ in the double summation. This leads to the following explicit recursive expression for $w^{(p)}$ at the order $O(\mu^{2p})$:

$$w^{(p)}=\frac{(-1)^{p+1}}{(2p+1)!}\nabla\cdot[h^{2p+1}\nabla(\nabla^{2p-2}(\nabla\cdot\hat{\boldsymbol{u}}))]$$
$$+\sum_{n=0}^{p-1}\frac{(-1)^n}{(2n+2)!}\nabla\cdot[h^{2n+2}\nabla(\nabla^{2n}w^{(p-n-1)})]. \tag{2.13}$$

We note that (2.13) and (2.12) is a simple recipe to express $\hat{w}$ in terms of $\hat{\boldsymbol{u}}$ to any order in μ. Here we include the first three terms (i.e. $p=0,1,2$) and derive the following explicit relation for $\hat{w}$ in terms of $\hat{\boldsymbol{u}}$:

$$\begin{aligned}\hat{w}(x,y,t)=&-\nabla\cdot(h\hat{\boldsymbol{u}})+\mu^2\nabla\cdot(\tfrac{1}{6}h^3\nabla(\nabla\cdot\hat{\boldsymbol{u}})-\tfrac{1}{2}h^2\nabla(\nabla\cdot(h\hat{\boldsymbol{u}})))\\&+\mu^4\nabla\cdot(\tfrac{1}{24}h^4\nabla(\nabla^2(\nabla\cdot(h\hat{\boldsymbol{u}})))-\tfrac{1}{120}h^5\nabla(\nabla^2(\nabla\cdot\hat{\boldsymbol{u}}))\\&+\tfrac{1}{2}h^2\nabla(\nabla\cdot(\tfrac{1}{6}h^3\nabla(\nabla\cdot\hat{\boldsymbol{u}})-\tfrac{1}{2}h^2\nabla(\nabla\cdot(h\hat{\boldsymbol{u}})))))+O(\mu^6).\end{aligned} \tag{2.14}$$

The combination of (2.14) and (2.8) provides a description of the wave kinematics in terms of derivatives of $\hat{\boldsymbol{u}}$ to the order $O(\mu^6)$.

(d) Boussinesq equations expressed in terms of $\hat{\boldsymbol{u}}$

The traditional approach in the derivation of Boussinesq-type equations is to integrate the horizontal momentum equations over depth. This requires a determination of the pressure distribution, which involves integration of the vertical momentum equation. As an alternative, Mei (1983) and Chen & Liu (1995) used the dynamic free surface boundary condition (2.3 *c*) and avoided the complications of the pressure calculation. The procedure is as follows. Use (2.7) to determine the velocity potential and its derivatives at the free surface and insert the result in (2.3 *c*) to obtain a Bernoulli equation in terms of $\Phi^{(0)}$ and $\Phi^{(1)}$. Apply the horizontal gradient operator and insert (2.9) to obtain a momentum vector equation in terms of $\hat{w}$ and $\hat{\boldsymbol{u}}$. Insert (2.14) to eliminate $\hat{w}$. For an arbitrary value of ϵ, this procedure leads to

$$\hat{\boldsymbol{u}}_t + \nabla\eta + \tfrac{1}{2}\epsilon\nabla(\hat{\boldsymbol{u}}^2) + \mu^2[\epsilon\Lambda_{21}^{\mathrm{I}} + \epsilon^2\Lambda_{22}^{\mathrm{I}} + \epsilon^3\Lambda_{23}^{\mathrm{I}}] + \mu^4[\epsilon\Lambda_{41}^{\mathrm{I}} + \epsilon^2\Lambda_{42}^{\mathrm{I}} + \epsilon^3\Lambda_{43}^{\mathrm{I}} + \epsilon^4\Lambda_{44}^{\mathrm{I}} + \epsilon^5\Lambda_{45}^{\mathrm{I}}] = O(\mu^6), \quad (2.15)$$

where

$$\Lambda_{21}^{\mathrm{I}} = \nabla[-\eta\nabla\cdot(h\hat{\boldsymbol{u}}_t) + \tfrac{1}{2}(\nabla\cdot(h\hat{\boldsymbol{u}}))^2], \quad (2.16\,a)$$

$$\Lambda_{22}^{\mathrm{I}} = \nabla[-\tfrac{1}{2}\eta^2\nabla\cdot\hat{\boldsymbol{u}}_t - \eta\hat{\boldsymbol{u}}\cdot\nabla(\nabla\cdot(h\hat{\boldsymbol{u}})) + \eta(\nabla\cdot\hat{\boldsymbol{u}})(\nabla\cdot(h\hat{\boldsymbol{u}}))], \quad (2.16\,b)$$

$$\Lambda_{23}^{\mathrm{I}} = \nabla[-\tfrac{1}{2}\eta^2\hat{\boldsymbol{u}}\cdot\nabla(\nabla\cdot\hat{\boldsymbol{u}}) + \tfrac{1}{2}\eta^2(\nabla\cdot\hat{\boldsymbol{u}})^2], \quad (2.16\,c)$$

$$\Lambda_{41}^{\mathrm{I}} = \nabla[\eta(\nabla\cdot(h^2\hat{\Gamma}_t)) - (\nabla\cdot(h^2\hat{\Gamma}))\nabla\cdot(h\hat{\boldsymbol{u}})], \quad (2.16\,d)$$

$$\Lambda_{42}^{\mathrm{I}} = \nabla[\eta\hat{\boldsymbol{u}}\cdot\nabla(\nabla\cdot(h^2\hat{\Gamma})) - \eta(\nabla\cdot\hat{\boldsymbol{u}})\nabla\cdot(h^2\hat{\Gamma})], \quad (2.16\,e)$$

$$\Lambda_{43}^{\mathrm{I}} = \nabla[\tfrac{1}{6}\eta^3\nabla^2(\nabla\cdot(h\hat{\boldsymbol{u}}_t)) + \tfrac{1}{2}\eta(\nabla(\nabla\cdot(h\hat{\boldsymbol{u}})))^2 - \tfrac{1}{2}\eta^2\nabla\cdot(h\hat{\boldsymbol{u}})\nabla^2(\nabla\cdot(h\hat{\boldsymbol{u}}))], \quad (2.16\,f)$$

$$\Lambda_{44}^{\mathrm{I}} = \nabla[\tfrac{1}{2}\eta^3\nabla(\nabla\cdot\hat{\boldsymbol{u}})\cdot\nabla(\nabla\cdot(h\hat{\boldsymbol{u}})) + \tfrac{1}{6}\eta^3\hat{\boldsymbol{u}}\cdot\nabla(\nabla^2(\nabla\cdot(h\hat{\boldsymbol{u}}))) - \tfrac{1}{2}\eta^3(\nabla\cdot\hat{\boldsymbol{u}})\nabla^2(\nabla\cdot(h\hat{\boldsymbol{u}})) - \tfrac{1}{6}\eta^3\nabla\cdot(h\hat{\boldsymbol{u}})\nabla^2(\nabla\cdot\hat{\boldsymbol{u}}) + \tfrac{1}{24}\eta^4\nabla^2(\nabla\cdot\hat{\boldsymbol{u}}_t)], \quad (2.16\,g)$$

$$\Lambda_{45}^{\mathrm{I}} = \nabla[\tfrac{1}{8}\eta^4(\nabla(\nabla\cdot\hat{\boldsymbol{u}}))^2 + \tfrac{1}{24}\eta^4\hat{\boldsymbol{u}}\cdot\nabla(\nabla^2(\nabla\cdot\hat{\boldsymbol{u}})) - \tfrac{1}{6}\eta^4(\nabla\cdot\hat{\boldsymbol{u}})\nabla^2(\nabla\cdot\hat{\boldsymbol{u}})], \quad (2.16\,h)$$

and where

$$\hat{\Gamma} \equiv \tfrac{1}{6}h\nabla(\nabla\cdot\hat{\boldsymbol{u}}) - \tfrac{1}{2}\nabla(\nabla\cdot(h\hat{\boldsymbol{u}})). \quad (2.17)$$

In traditional Boussinesq theory, the leading orders of nonlinearity and dispersion are assumed to be equal, i.e. $\epsilon = O(\mu^2)$. With this assumption, higher-order Boussinesq theory would neglect, for example, $O(\epsilon^2\mu^2)$ and $O(\epsilon\mu^4)$ which would then be comparable to $O(\mu^6)$. In fact, disregarding these terms in (2.15), the one-dimensional form of the result reduces to the higher-order Boussinesq equation (D39) of Dingemans (1973).

To close the system of equations we need to satisfy either the remaining kinematic boundary condition (2.3 *d*) at the free surface or equivalently, the depth-integrated continuity equation,

$$\eta_t + \nabla\cdot\boldsymbol{Q} = 0, \quad \boldsymbol{Q} \equiv \int_{-h}^{\epsilon\eta} \nabla\Phi\,\mathrm{d}z. \quad (2.18)$$

Here we follow the latter approach, and insert (2.7), (2.9) and (2.14) in (2.18) which after integration leads to

$$\boldsymbol{Q} = \{\hat{\boldsymbol{u}}(h+\epsilon\eta) - \mu^2[\tfrac{1}{6}(\epsilon^3\eta^3 + h^3)\nabla(\nabla\cdot\hat{\boldsymbol{u}}) + \tfrac{1}{2}(\epsilon^2\eta^2 - h^2)\nabla(\nabla\cdot(h\hat{\boldsymbol{u}}))] + \mu^4[\tfrac{1}{2}(\epsilon^2\eta^2 - h^2)\nabla(\nabla\cdot(h^2\hat{\Gamma})) + \tfrac{1}{24}(\epsilon^4\eta^4 - h^4)\nabla(\nabla^2(\nabla\cdot(h\hat{\boldsymbol{u}}))) + \tfrac{1}{120}(\epsilon^5\eta^5 + h^5)\nabla(\nabla^2(\nabla\cdot\hat{\boldsymbol{u}}))]\} + O(\mu^6), \quad (2.19)$$

where $\hat{\Gamma}$ is defined by (2.17). The combination of (2.15) and (2.18) with (2.19) represents a complete system of higher-order Boussinesq-type equations truncated at $O(\mu^6)$ and retaining all nonlinear terms. If terms of order $O(\epsilon^2\mu^2, \epsilon\mu^4)$ are neglected, then the one-dimensional form reduces to (D39) in Dingemans (1973).

It is emphasized that the equations derived above in terms of $\hat{\boldsymbol{u}}$ serve mainly as a starting point for recasting the equations in terms of other velocity variables, such as the depth-averaged velocity, $\boldsymbol{U}$ (§§ 3 and 4), and the velocity at an arbitrary z location, $\tilde{\boldsymbol{u}}$ (§§ 5 and 6). The $\hat{\boldsymbol{u}}$ formulation will reappear as a subset of the more general formulation in § 5, and for this reason an analysis of the equations is postponed to § 5 *b*.

3. Equations in terms of the depth-averaged velocity, $\boldsymbol{U}$

(*a*) *Derivation of* $(\mu^4, \epsilon\mu^4)$ *equations*

In this and the following section we shall consider formulations in terms of the depth-averaged velocity vector, $\boldsymbol{U}$, and analyse the resulting equations with respect to dispersion and nonlinearity. The depth-averaged velocity is defined by

$$\boldsymbol{U} \equiv \boldsymbol{Q}/(h+\epsilon\eta), \tag{3.1}$$

and one of the advantages of using this variable is that the continuity equation becomes exact and relatively simple,

$$\eta_t + \nabla\cdot((h+\epsilon\eta)\boldsymbol{U}) = 0. \tag{3.2}$$

This is obviously attractive compared to the complicated form of (2.19) in terms of the velocity at the still water level, $\hat{\boldsymbol{u}}$.

In order to derive higher-order momentum equations in terms of $\boldsymbol{U}$, we shall use (2.15) in terms of $\hat{\boldsymbol{u}}$ as a starting point. The first step is then to establish a relation between $\boldsymbol{U}$ and $\hat{\boldsymbol{u}}$, and this is obtained by inserting (2.19) in (3.1), which yields

$$\begin{aligned}\boldsymbol{U} = \hat{\boldsymbol{u}} &+ \mu^2[\tfrac{1}{2}(h-\epsilon\eta)\nabla(\nabla\cdot(h\hat{\boldsymbol{u}})) - \tfrac{1}{6}(h^2-\epsilon h\eta+\epsilon^2\eta^2)\nabla(\nabla\cdot\hat{\boldsymbol{u}})] \\ &+ \mu^4[-\tfrac{1}{2}(h-\epsilon\eta)\nabla(\nabla\cdot(h^2\hat{\Gamma})) - \tfrac{1}{24}(h^3-\epsilon\eta h^2+\epsilon^2\eta^2 h-\epsilon^3\eta^3)\nabla(\nabla^2(\nabla\cdot(h\hat{\boldsymbol{u}}))) \\ &+ \tfrac{1}{120}(h^4-\epsilon\eta h^3+\epsilon^2\eta^2h^2-\epsilon^3\eta^3h+\epsilon^4\eta^4)\nabla(\nabla^2(\nabla\cdot\hat{\boldsymbol{u}}))] + O(\mu^6).\end{aligned} \tag{3.3}$$

To replace $\hat{\boldsymbol{u}}$ by $\boldsymbol{U}$ in the momentum equation we need to establish the inverse relation of (3.3) in which $\hat{\boldsymbol{u}}$ is expressed in terms of $\boldsymbol{U}$. This is achieved by the use of successive substitutions starting at lowest order in μ^2 and results in

$$\begin{aligned}\hat{\boldsymbol{u}} = \boldsymbol{U} &- \mu^2[\tfrac{1}{2}(h-\epsilon\eta)\nabla(\nabla\cdot(h\boldsymbol{U})) - \tfrac{1}{6}(h^2-\epsilon\eta h+\epsilon^2\eta^2)\nabla(\nabla\cdot\boldsymbol{U})] \\ &+ \mu^4[\tfrac{1}{24}(h^3-\epsilon\eta h^2+\epsilon^2\eta^2h-\epsilon^3\eta^3)\nabla(\nabla^2(\nabla\cdot(h\boldsymbol{U}))) \\ &- \tfrac{1}{120}(h^4-\epsilon\eta h^3+\epsilon^2\eta^2h^2-\epsilon^3\eta^3h+\epsilon^4\eta^4)\nabla(\nabla^2(\nabla\cdot\boldsymbol{U})) \\ &- \tfrac{1}{2}(h-\epsilon\eta)\nabla(\nabla\cdot(\tfrac{1}{2}\epsilon\eta h\nabla(\nabla\cdot(h\boldsymbol{U})) - \tfrac{1}{6}(\epsilon\eta h^2-\epsilon^2\eta^2h)\nabla(\nabla\cdot\boldsymbol{U}))) \\ &- \tfrac{1}{6}(h^2-\epsilon\eta h+\epsilon^2\eta^2)\nabla(\nabla\cdot(\tfrac{1}{2}(h-\epsilon\eta)\nabla(\nabla\cdot(h\boldsymbol{U})) \\ &- \tfrac{1}{6}(h^2-\epsilon\eta h+\epsilon^2\eta^2)\nabla(\nabla\cdot\boldsymbol{U})))] + O(\mu^6).\end{aligned} \tag{3.4}$$

The next step is to insert (3.4) in (2.15) and (2.16), which yields a fully nonlinear momentum equation in terms of the depth-averaged velocity,

$$\begin{aligned}\boldsymbol{U}_t + \nabla\eta + \tfrac{1}{2}\epsilon\nabla(\boldsymbol{U}^2) &+ \mu^2(\Lambda_{20}^{\mathrm{II}} + \epsilon\Lambda_{21}^{\mathrm{II}} + \epsilon^2\Lambda_{22}^{\mathrm{II}} + \epsilon^3\Lambda_{23}^{\mathrm{II}}) \\ &+ \mu^4(\Lambda_{40}^{\mathrm{II}} + \epsilon\Lambda_{41}^{\mathrm{II}} + \epsilon^2\Lambda_{42}^{\mathrm{II}} + \epsilon^3\Lambda_{43}^{\mathrm{II}} + \epsilon^4\Lambda_{44}^{\mathrm{II}} + \epsilon^5\Lambda_{45}^{\mathrm{II}}) = O(\mu^6).\end{aligned} \tag{3.5}$$

It turns out that the complexity of especially the terms $\Lambda_{41}^{\mathrm{II}}$, $\Lambda_{42}^{\mathrm{II}}$, $\Lambda_{43}^{\mathrm{II}}$, $\Lambda_{44}^{\mathrm{II}}$ and $\Lambda_{45}^{\mathrm{II}}$ is considerable and in order to simplify the equations we introduce a relation between the parameters ϵ and μ, which up to now have only served to denote nonlinear terms or dispersive terms. While the classical Boussinesq equations are based on the assumption of $\epsilon = O(\mu^2)$, we shall allow the nonlinearity to be stronger and consider the case of

$$\epsilon = O(\mu), \tag{3.6 a}$$

by which the terms $\Lambda_{42}^{\mathrm{II}}$, $\Lambda_{43}^{\mathrm{II}}$, $\Lambda_{44}^{\mathrm{II}}$ and $\Lambda_{45}^{\mathrm{II}}$ can be ignored in (3.5). As an additional assumption we consider a mildly sloping bottom and assume that

$$|\nabla^n h| = O(\mu^n), \quad n = 1, 2, 3, \ldots, \tag{3.6 b}$$

by which the spatial variation of the depth can be ignored in $\Lambda_{41}^{\mathrm{II}}$. Now the momentum equation (3.5) simplifies to

$$\boldsymbol{U}_t + \nabla\eta + \tfrac{1}{2}\epsilon\nabla(\boldsymbol{U}^2) + \mu^2(\Lambda_{20}^{\mathrm{II}} + \epsilon\Lambda_{21}^{\mathrm{II}} + \epsilon^2\Lambda_{22}^{\mathrm{II}} + \epsilon^3\Lambda_{23}^{\mathrm{II}}) + \mu^4(\Lambda_{40}^{\mathrm{II}} + \epsilon\Lambda_{41}^{\mathrm{II}}) = O(\mu^6, \epsilon^2\mu^4), \tag{3.7}$$

where

$$\Lambda_{20}^{\mathrm{II}} = h\Gamma_t, \tag{3.8 a}$$

$$\Lambda_{21}^{\mathrm{II}} = -\eta\Gamma_t + \nabla\cdot(h\boldsymbol{U})\Gamma + \nabla(\boldsymbol{U}\cdot(h\Gamma) - \eta\nabla\cdot(h\boldsymbol{U}_t) + \tfrac{1}{2}(\nabla\cdot(h\boldsymbol{U}))^2), \tag{3.8 b}$$

$$\begin{aligned}\Lambda_{22}^{\mathrm{II}} = {} & \tfrac{1}{6}\eta^2\nabla(\nabla\cdot\boldsymbol{U}_t) - \tfrac{1}{3}\eta\nabla\cdot(h\boldsymbol{U})\nabla(\nabla\cdot\boldsymbol{U}) + \nabla\cdot(\eta\boldsymbol{U})\Gamma \\ & + \nabla(-\eta\boldsymbol{U}\cdot\Gamma - \tfrac{1}{2}\eta^2\nabla\cdot\boldsymbol{U}_t + \eta(\nabla\cdot\boldsymbol{U})\nabla\cdot(h\boldsymbol{U}) - \eta\boldsymbol{U}\cdot\nabla(\nabla\cdot(h\boldsymbol{U}))),\end{aligned} \tag{3.8 c}$$

$$\Lambda_{23}^{\mathrm{II}} = -\tfrac{1}{3}\eta\nabla\cdot(\eta\boldsymbol{U})\nabla(\nabla\cdot\boldsymbol{U}) + \nabla(-\tfrac{1}{3}\eta^2\boldsymbol{U}\cdot\nabla(\nabla\cdot\boldsymbol{U}) + \tfrac{1}{2}\eta^2(\nabla\cdot\boldsymbol{U})^2), \tag{3.8 d}$$

$$\Lambda_{40}^{\mathrm{II}} = \tfrac{1}{24}h^3\nabla(\nabla^2(\nabla\cdot(h\boldsymbol{U}_t))) - \tfrac{1}{120}h^4\nabla(\nabla^2(\nabla\cdot\boldsymbol{U}_t)) + \tfrac{1}{6}h^2\nabla(\nabla\cdot(h\Gamma_t)), \tag{3.8 e}$$

$$\begin{aligned}\Lambda_{41}^{\mathrm{II}} = {} & \tfrac{1}{45}h^3\eta\nabla(\nabla^2(\nabla\cdot\boldsymbol{U}_t)) - \tfrac{1}{9}h^3\nabla(\nabla\cdot(\eta\nabla(\nabla\cdot\boldsymbol{U}_t))) - \tfrac{1}{45}h^4\nabla\cdot\boldsymbol{U}\nabla(\nabla^2(\nabla\cdot\boldsymbol{U})) \\ & + \tfrac{1}{9}h^4\nabla(\nabla\cdot(\nabla\cdot\boldsymbol{U}(\nabla(\nabla\cdot\boldsymbol{U})))) + \tfrac{1}{18}h^4\nabla(\nabla(\nabla\cdot\boldsymbol{U}))^2 \\ & - \tfrac{1}{45}h^4\nabla(\boldsymbol{U}\cdot\nabla(\nabla^2(\nabla\cdot\boldsymbol{U}))) + O(\mu),\end{aligned} \tag{3.8 f}$$

and where

$$\Gamma = \tfrac{1}{6}h\nabla(\nabla\cdot\boldsymbol{U}) - \tfrac{1}{2}\nabla(\nabla\cdot(h\boldsymbol{U})). \tag{3.8 g}$$

In the derivation of (3.8) we have eliminated η_t by the use of the continuity equation. In fact, (3.6 *b*) allows for a further simplification of (3.8 *e*), where only first derivatives of h are included, i.e.

$$\Lambda_{40}^{\mathrm{II}} = [-\tfrac{1}{45}h^4\nabla(\nabla^2(\nabla\cdot\boldsymbol{U}_t)) - \tfrac{2}{9}h^3\nabla h\nabla^2(\nabla\cdot\boldsymbol{U}_t)] + O(\mu^2). \tag{3.9}$$

Notice that if we disregard the $\epsilon^2\mu^2$,$\epsilon^3\mu^2$ and $\epsilon\mu^4$ terms, i.e. (3.8 *c*), (3.8 *d*) and (3.8 *f*), the one-dimensional form of the result reduces to the higher-order Boussinesq equation (D38) by Dingemans (1973).

(*b*) *Fourier analysis of equations on a horizontal bottom*

In this subsection we shall analyse equations (3.2) and (3.7) to quantify the embedded characteristics with respect to dispersion and nonlinearity. Although the derivation of the equations has been based on the assumption of $\mu \ll 1$ and $\epsilon = O(\mu)$,

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the analysis will be made under the assumption of $\epsilon \ll 1$ (i.e. weakly nonlinear solutions) and arbitrary μ. We shall make a Stokes-type Fourier analysis on a horizontal bottom and look for first-, second- and third-order solutions of the form,

$$\eta = a_1 \cos\theta + \epsilon a_2 \cos 2\theta + \epsilon^2 a_3 \cos 3\theta, \quad U = U_1 \cos\theta + \epsilon U_2 \cos 2\theta + \epsilon^2 U_3 \cos 3\theta, \tag{3.10}$$

where $\theta = \omega t - kx$. Hence it is emphasized that the analysis will not involve nonlinear terms with powers of ϵ higher than two, although such terms are retained in the formulation. In one dimension, and on a horizontal bottom, (3.2) and (3.7) simplify to

$$\eta_t + hU_x + \epsilon(\eta U)_x = 0, \qquad (3.11\,a)$$

$$\begin{aligned} U_t + \eta_x - \tfrac{1}{3}\mu^2 h^2 U_{xxt} - \tfrac{1}{45}\mu^4 h^4 U_{xxxxt} + \epsilon U U_x \\ + \epsilon\mu^2(-\tfrac{2}{3}h\eta U_{xxt} - h\eta_x U_{xt} + \tfrac{1}{3}h^2 U_x U_{xx} - \tfrac{1}{3}h^2 U U_{xxx}) \\ + \epsilon\mu^4(\tfrac{1}{45}h^3\eta U_{xxxxt} - \tfrac{1}{9}h^3(\eta U_{xxt})_{xx} - \tfrac{1}{45}h^4 U_x U_{xxxx} \\ + \tfrac{1}{9}h^4(U_x U_{xx})_{xx} + \tfrac{1}{9}h^4 U_{xx}U_{xxx} - \tfrac{1}{45}h^4(U U_{xxxx})_x) \\ + \epsilon^2\mu^2(\tfrac{1}{6}\eta^2 U_{xxt} - \tfrac{1}{3}\eta h U_x U_{xx} - \tfrac{1}{3}h U_{xx}(\eta U)_x + h(\eta U_x^2)_x \\ - \tfrac{1}{2}(\eta^2 U_{xt})_x - \tfrac{2}{3}h(\eta U U_{xx})_x) = O(\epsilon^3\mu^2, \epsilon^2\mu^4, \mu^6). \end{aligned} \qquad (3.11\,b)$$

In the following analysis, which is performed directly in non-dimensional variables, it is convenient to introduce the definition

$$\kappa \equiv \mu k h, \tag{3.12}$$

and we notice that κ is actually identical to $k'h'$, which is the product of the dimensional wavenumber and depth.

(i) *First-order solution*

By substituting (3.10) into (3.11), we get at the order $O(\epsilon^0)$

$$\begin{pmatrix} m_{11}^{(1)} & m_{12}^{(1)} \\ m_{21}^{(1)} & m_{22}^{(1)} \end{pmatrix} \begin{pmatrix} a_1 \\ U_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{3.13}$$

where

$$m_{11}^{(1)} = \omega, \quad m_{12}^{(1)} = -kh, \quad m_{21}^{(1)} = -k, \quad m_{22}^{(1)} = \omega(1 + \tfrac{1}{3}\kappa^2 - \tfrac{1}{45}\kappa^4). \tag{3.14}$$

Hence, at first order we get

$$U_1 = \omega a_1/(kh) \qquad (3.15\,a)$$

and the dispersion relation

$$\frac{\omega^2}{k^2 h} = \frac{1}{1 + \frac{1}{3}\kappa^2 - \frac{1}{45}\kappa^4}, \qquad (3.15\,b)$$

while the solution corresponding to classical lower-order Boussinesq equations is found by ignoring the κ^4 term in (3.15 *b*). In this context the reference solution is the linear dispersion relation of Stokes,

$$\left(\frac{\omega^2}{k^2 h}\right)^{\text{Stokes}} = \frac{\tanh(\kappa)}{\kappa}. \tag{3.16}$$

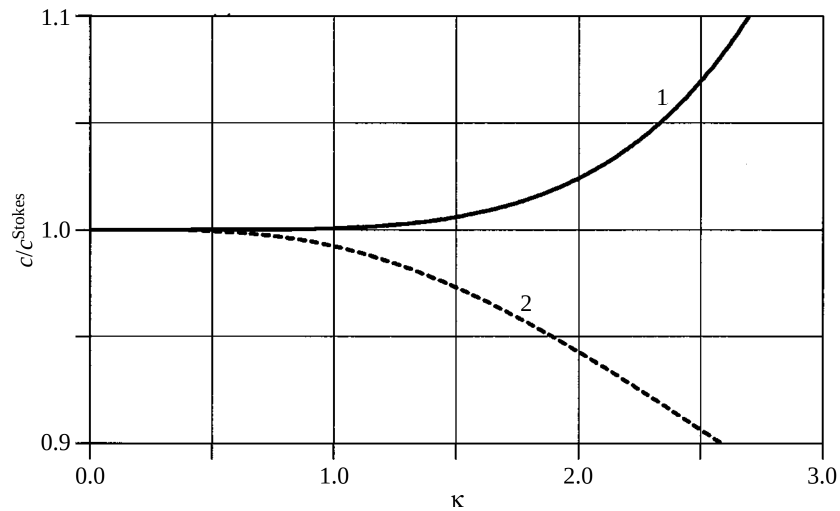

Figure 1. Ratio of phase celerity, c/c^{Stokes}, where c is determined by (3.15 b) and c^{Stokes} by (3.16). Boussinesq equations include (1) $O(\mu^4)$, Padé [0,4]; (2) $O(\mu^2)$, Padé [0,2].

The expression (3.15 b) is a Padé [0,4] expansion in κ of (3.16). For later reference we note that a Padé [4,4] expansion of (3.16) reads

$$\left(\frac{\omega^2}{k^2h}\right)^{\text{Stokes}} = \frac{1+\frac{1}{9}\kappa^2+\frac{1}{945}\kappa^4}{1+\frac{4}{9}\kappa^2+\frac{1}{63}\kappa^4} + O(\kappa^{10}). \tag{3.17}$$

The ratio of phase celerity, c/c^{Stokes}, where c is determined from (3.15 b) and c^{Stokes} by (3.16), is shown in figure 1. As expected, we notice an improved accuracy for small κ values in comparison with the lower-order equations. Unfortunately, a singularity occurs in (3.15 b) for $\kappa^2 = \frac{1}{2}(15+9\sqrt{5})$, i.e. $\kappa \approx 4.2$, and although this is a fairly large wavenumber this singularity turns out to be fatal for any practical use of the higher-order equations: numerical instabilities show up at this wavenumber even in otherwise calm water. In § 4 we shall derive an enhanced set of equations without this singularity.

(ii) *Second-order solution*

Continuing the present analysis to second order, we substitute (3.10) into (3.11) and collect terms of $O(\epsilon)$. This leads to

$$\begin{pmatrix} m_{11}^{(2)} & m_{12}^{(2)} \\ m_{21}^{(2)} & m_{22}^{(2)} \end{pmatrix} \begin{pmatrix} a_2 \\ U_2 \end{pmatrix} = \frac{a_1^2}{h}\begin{pmatrix} F_1 \\ F_2 \end{pmatrix}, \tag{3.18}$$

where

$$m_{11}^{(2)} = 2\omega, \quad m_{12}^{(2)} = -2kh, \quad m_{21}^{(2)} = -2k, \quad m_{22}^{(2)} = 2\omega(1+\tfrac{4}{3}\kappa^2-\tfrac{16}{45}\kappa^4), \tag{3.19 a}$$

$$F_1 = \omega, \quad F_2 = (\omega^2/2kh)(1-\tfrac{5}{3}\kappa^2+\tfrac{41}{45}\kappa^4). \tag{3.19 b}$$

Hence, at second order we get the solution

$$a_2 = \frac{a_1^2}{h}\left(\frac{F_1 m_{22}^{(2)} - F_2 m_{12}^{(2)}}{m_{11}^{(2)} m_{22}^{(2)} - m_{21}^{(2)} m_{12}^{(2)}}\right), \tag{3.20 a}$$

Table 1. *Expansion coefficients for the second harmonic*

	Stokes	$(\mu^4, \epsilon\mu^4)$	$(\mu^4, \epsilon\mu^2)$	$(\mu^2, \epsilon\mu^2)$	(μ^2, ϵ)
c_2	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{8}{9}$
c_4	$\frac{7}{45}$	$\frac{13}{45}$	$-\frac{2}{135}$		

$$U_2 = \frac{a_1^2}{h}\left(\frac{m_{11}^{(2)}F_2 - m_{21}^{(2)}F_1}{m_{11}^{(2)}m_{22}^{(2)} - m_{21}^{(2)}m_{12}^{(2)}}\right), \tag{3.20 b}$$

Here we shall focus on a_2 and the reference in this context is the Stokes second-order solution (see, for example, Skjelbreia & Hendrickson 1960),

$$a_2^{\text{Stokes}} = \frac{1}{4}\left(\frac{a_1^2}{h}\right)\kappa\coth(\kappa)(3\coth^2(\kappa) - 1), \tag{3.21}$$

of which an expansion from $\kappa = 0$ yields

$$a_2^{\text{Stokes}} = \frac{3}{4}\frac{a_1^2}{h}\frac{1}{\kappa^2}(1 + \tfrac{2}{3}\kappa^2 + \tfrac{7}{45}\kappa^4 + \tfrac{2}{315}\kappa^6 + O(\kappa^8)). \tag{3.22}$$

The expansion of the second-order Fourier solution (3.20 *a*) to the Boussinesq equations yields

$$a_2 = \frac{3}{4}\frac{a_1^2}{h}\frac{1}{\kappa^2}(1 + c_2\kappa^2 + c_4\kappa^4 + O(\kappa^6)), \tag{3.23}$$

where $c_2 = \frac{2}{3}$ and $c_4 = \frac{13}{45}$ when $(\mu^4, \epsilon\mu^4)$ terms are retained. If we retain only $(\mu^4, \epsilon\mu^2)$ terms, it leads to the neglect of the κ^4 term in (3.19 *b*) and results in $c_2 = \frac{2}{3}$ and $c_4 = -\frac{2}{135}$. Hence in both cases the c_4 coefficient is incorrect. At lower order we may retain $(\mu^2, \epsilon\mu^2)$ terms which leads to a further neglect of the κ^4 term in (3.19 *a*) and (3.15 *b*) and results in $c_2 = \frac{1}{3}$. If we retain only (μ^2, ϵ) terms, as in the classical equations of Peregrine (1967), it leads to a further neglect of the κ^2 term in (3.19 *b*) and results in $c_2 = \frac{8}{9}$. Hence in both cases only the leading order in (3.22) is matched. The results are summarized in table 1.

In order to explain the above results, we note that the expansion (3.22) is basically proportional to ϵ/μ^2 multiplied by a power series in μ^2. When the transfer function is derived on the basis of Boussinesq-type equations, the result will be determined by nonlinear terms of order ϵ in the numerator and linear terms in the denominator. The denominator for a given harmonic turns out to be proportional to the wavenumber mismatch between the bound and the free wavenumber of that harmonic. For this reason, the accuracy of the denominator always has to be a factor μ^2 higher than the accuracy of the numerator in order to obtain a certain accuracy in the matching with the Stokes target solution. This is confirmed by the fact that classical Boussinesq equations which retain $O(\mu^2, \epsilon)$ terms are able to match only the leading terms in the Stokes expansion, while a correct value of c_2 in (3.23) requires a combination of μ^4 and $\epsilon\mu^2$ terms. Similarly, a correct value of c_4 will require the inclusion of μ^6 and $\epsilon\mu^4$ terms in the Boussinesq formulation. A further discussion of the possibility of matching (3.22) on the basis of Boussinesq-type equations is given in § 4 *b*.

Figure 2 shows the ratio of a_2/a_2^{Stokes} (i.e. (3.20) and (3.21)) for the four different versions of the Boussinesq equations discussed above. We notice that the two μ^4

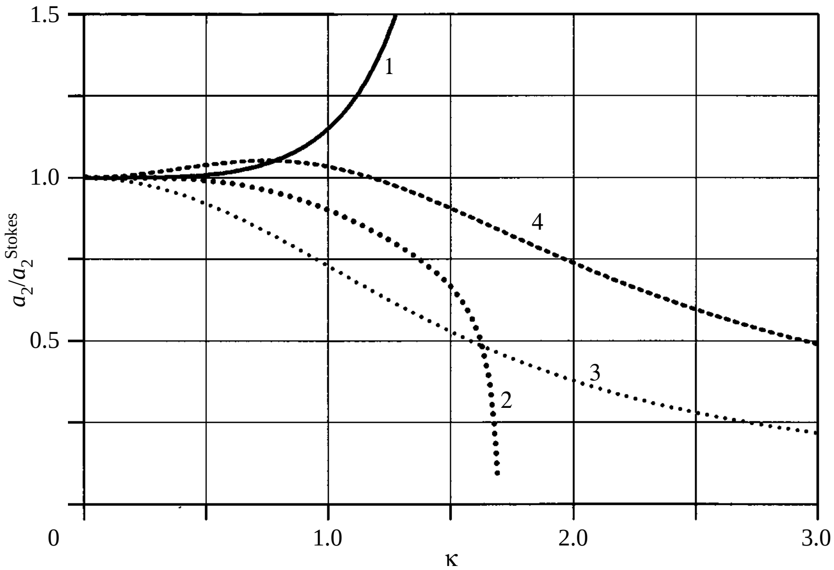

Figure 2. Ratio of second harmonic, a_2/a_2^{Stokes}, where a_2 is determined by (3.20 *a*) and a_2^{Stokes} by (3.21). Boussinesq equations include (1) $O(\mu^4, \epsilon\mu^4)$; (2) $O(\mu^4, \epsilon\mu^2)$; (3) $O(\mu^2, \epsilon\mu^2)$; (4) $O(\mu^2, \epsilon)$.

formulations are superior for $\kappa < 0.75$, whereas they are rather poor for larger values of κ. In fact, they both contain a singularity for $\kappa = \sqrt{3}$, which again makes these formulations useless for practical purposes. The singularity in a_2 will be removed in § 4.

(iii) *Third-order solution*

Continuing the analysis to third order, we insert (3.10) in (3.11) and collect terms of order $O(\epsilon^2)$. The terms proportional to $\sin 3\theta$ directly lead to a matrix problem similar to (3.13) and (3.18) which results in third-order solutions for a_3 and U_3. In addition to this problem terms proportional to $\sin\theta$ need to be removed to avoid secular unbounded solutions. This problem is solved by expanding ω and U_1 as

$$\omega = \omega_1(1 + \epsilon^2\omega_{13}), \quad U_1 = (\omega a_1/kh)(1 + \epsilon^2 U_{13}), \tag{3.24}$$

where ω_{13} represents the amplitude dispersion and U_{13} the third-order correction to the first-order velocity amplitude. Here we shall concentrate on the quantities a_3 and ω_{13} and the reference in this context is the Stokes third-order solution (see, for example, Skjelbreia & Hendrickson 1960),

$$a_3^{\text{Stokes}} = \frac{3}{64}\frac{a_1^3}{h^2}\kappa^2\frac{1 + 8\cosh^6\kappa}{\sinh^6\kappa}, \quad \omega_{13}^{\text{Stokes}} = \frac{1}{16}\frac{a_1^2}{h^2}\kappa^2\frac{9\tanh^4\kappa - 10\tanh^2\kappa + 9}{\tanh^4\kappa}, \tag{3.25}$$

of which an expansion from $\kappa = 0$ reads

$$\left.\begin{aligned} a_3^{\text{Stokes}} &= \frac{27}{64}\left(\frac{a_1^3}{h^2}\right)\left(\frac{1}{\kappa^4}\right)(1 + \tfrac{5}{3}\kappa^2 + \tfrac{64}{45}\kappa^4 + \tfrac{85}{189}\kappa^6 + O(\kappa^8)), \\ \omega_{13}^{\text{Stokes}} &= \frac{9}{16}\left(\frac{a_1^2}{h^2}\right)\left(\frac{1}{\kappa^2}\right)(1 + \tfrac{2}{9}\kappa^2 + \tfrac{113}{135}\kappa^4 - \tfrac{2}{315}\kappa^6 + O(\kappa^8)). \end{aligned}\right\} \tag{3.26}$$

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The Boussinesq equations (3.11) result in expansions similar to (3.26) and the following conclusion can be made: only the leading-order expressions in (3.26) are matched if we retain (μ^2, ϵ) terms in (3.11). If we retain $(\mu^4, \epsilon\mu^2, \epsilon^2\mu^2)$ terms in (3.11), the coefficients of the κ^2 terms in (3.26) are matched. It turns out that the μ^4 Boussinesq formulation results in singularities, in this case occurring in a_3 and ω_{13} at $\kappa = \sqrt{3}$ and $\kappa = \sqrt{3}/\sqrt{2}$. These singularities will be removed in § 4.

4. Enhanced equations in terms of the depth-averaged velocity, *U*

(*a*) *Introduction*

In this section we shall generalize the enhancement technique introduced by Madsen *et al*. (1991) and Schäffer & Madsen (1995*a*) and apply it to the equations derived in § 3 with the following objectives. First of all, the technique significantly improves the linear dispersion characteristics embedded in the equations. In the present case this has the important side effect of removing all the singularities found in § 3 *b*, thus making a set of useless equations highly applicable. Second, we shall demonstrate that the enhancements carry over to the nonlinear characteristics of the equations, leading to better accuracy of higher harmonic transfers and amplitude dispersion. In later sections (§§ 7 and 8), we shall demonstrate that the enhancements further improves the sub- and superharmonic transfers and the linear dispersion characteristics in connection with waves in ambient currents. Recent numerical simulations (published by Madsen *et al*. 1996) have confirmed the quality of the new enhanced equations, when applied to phase-resolving problems.

The rationale of the enhancement technique is summarized in the following for the case of a horizontal bottom. First, we introduce the linear operator

$$L \equiv 1 + \sum_{n=1}^{M} \nu_n \mu^{2n} h^{2n} \nabla^{2n}, \quad \nu_n = O(1),$$

and apply it to those of the governing equations containing a remainder of $O(\mu^{2M})$. This operator has certain free parameters, ν_n. For a suitable choice of these parameters, terms of very high order vanish in the governing equations, thus resulting in equations which are of higher order than they appear. The enhanced equations can be achieved in two different but equivalent ways.

(i) *Method 1*

The first method requires that $M = 2N$ and that μ^{4N} is the highest order retained in the governing equations. After the use of the linear operator the resulting equation is still accurate to $O(\mu^{4N})$, but the coefficient of each term now depends on the parameters, ν_n. These M parameters are chosen so that the coefficients vanish for the M linear terms $O(\mu^{4N})$, $O(\mu^{4N-1})$, … $O(\mu^{2N+1})$. With this choice we obtain an equation in which only terms up to $O(\mu^{2N})$ show, but the accuracy is still as high as $O(\mu^{4N})$. The general rationale of this technique is thus to reduce the equation by extinguishing the highest-order linear terms without loss of accuracy. Only half of the parameters, ν_n, show in the final equation. Unfortunately, this method of determining the parameters ν_n is quite tedious, since the original equation has to be developed to $O(\mu^{4N})$, although only terms of $O(\mu^{2N})$ show in the final result.

(ii) *Method 2*

The second method, which leads to the same enhanced equations as the first one, involves less algebra as it only determines the parameters which appear explicitly in the final equation. In this case we use $M = N$, start with an order μ^{2N} equation, apply an order μ^{2N} operator, and determine the parameters by matching the dispersion relation embedded in the resulting equations with the result from fully dispersive theory. With this approach the accuracy of the equation is formally only shown to be $O(\mu^{2N})$ although it is identical to the result of accuracy $O(\mu^{4N})$ obtained by Method 1.

In the present paper we pursue the case of $N = 2$ by using the second approach. The equivalence of the two methods is shown for the case of $N = 1$. In § 4 *b* the different parts of the operator are applied to the momentum equation separately and due to the variable depth two components are present at each order in μ^2.

(*b*) *Derivation of* $(\mu^4, \epsilon\mu^4)$ *equations*

We introduce four free parameters $(\alpha_1, \alpha_2, \beta_1, \beta_2)$ of order $O(1)$. For a constant depth, two of these parameters (α_1 and β_1) would suffice. The other two govern the bottom slope terms, which again determine the shoaling properties of the resulting equations.

The first step in the procedure is to apply the operator $\nabla(\nabla\cdot)$ to (3.7) and multiply the result by $(\alpha_2 - \alpha_1)\mu^2 h^2$, which yields

$$(\alpha_2 - \alpha_1)\mu^2 h^2\{\nabla(\nabla\cdot\boldsymbol{U}_t) + \nabla(\nabla^2\eta) + \tfrac{1}{2}\epsilon\nabla(\nabla^2(\boldsymbol{U}^2)) + \mu^2\nabla(\nabla\cdot\Lambda^{\mathrm{II}}_{20}) + \epsilon\mu^2\nabla(\nabla\cdot\Lambda^{\mathrm{II}}_{21})\} = O(\epsilon^2\mu^4, \mu^6). \quad (4.1\,a)$$

The second step is to multiply (3.7) by h, apply the operator $\nabla(\nabla\cdot)$ and multiply the result by $-\alpha_2\mu^2 h$, which yields

$$-\alpha_2\mu^2 h\{\nabla(\nabla\cdot(h\boldsymbol{U}_t)) + \nabla(\nabla\cdot(h\nabla\eta)) + \tfrac{1}{2}\epsilon\nabla(\nabla\cdot(h\nabla(\boldsymbol{U}^2))) + \mu^2\nabla(\nabla\cdot(h\Lambda^{\mathrm{II}}_{20})) + \epsilon\mu^2\nabla(\nabla\cdot(h\Lambda^{\mathrm{II}}_{21}))\} = O(\epsilon^2\mu^4, \mu^6). \quad (4.1\,b)$$

The third step is to apply the operator $\nabla(\nabla^2(\nabla\cdot))$ to (3.7) and multiply the result by $\beta_1\mu^4 h^4$, which yields

$$\beta_1\mu^4 h^4\{\nabla(\nabla^2(\nabla\cdot\boldsymbol{U}_t)) + \nabla(\nabla^4\eta) + \tfrac{1}{2}\epsilon\nabla(\nabla^4(\boldsymbol{U}^2))\} = O(\mu^6). \quad (4.1\,c)$$

The fourth step is to apply the operator $\nabla^2(\nabla\cdot)$ to (3.7) and multiply the result by $\beta_2\mu^4 h^3\nabla h$, which yields

$$\beta_2\mu^4 h^3\nabla h\{\nabla^2(\nabla\cdot\boldsymbol{U}_t) + \nabla^4\eta\} = O(\mu^6). \quad (4.1\,d)$$

Each of the equations (4.1) are correct up to the same order as the original equation (3.7), which consequently can be consistently modified by the use of these equations. Finally, by adding (4.1) to (3.7) we obtain a new alternative higher-order momentum equation,

$$\boldsymbol{U}_t + \nabla\eta + \tfrac{1}{2}\epsilon\nabla(\boldsymbol{U}^2) + \mu^2(\Lambda^{\mathrm{III}}_{20} + \epsilon\Lambda^{\mathrm{III}}_{21} + \epsilon^2\Lambda^{\mathrm{III}}_{22} + \epsilon^3\Lambda^{\mathrm{III}}_{23}) + \mu^4(\Lambda^{\mathrm{III}}_{40} + \epsilon\Lambda^{\mathrm{III}}_{41}) = O(\epsilon^2\mu^4, \mu^6), \quad (4.2)$$

where

$$\Lambda_{20}^{\rm III} = [h^2(\tfrac{1}{6}+\alpha_2-\alpha_1)\nabla(\nabla\cdot\boldsymbol{U}_t) - h(\tfrac{1}{2}+\alpha_2)\nabla(\nabla\cdot(h\boldsymbol{U}_t)) + h^2(\alpha_2-\alpha_1)\nabla(\nabla^2\eta) - h\alpha_2\nabla(\nabla\cdot(h\nabla\eta))], \tag{4.3 a}$$

$$\Lambda_{21}^{\rm III} = [\tfrac{1}{2}h^2(\alpha_2-\alpha_1)\nabla(\nabla^2(\boldsymbol{U}^2)) - \tfrac{1}{2}h\alpha_2\nabla(\nabla\cdot(h\nabla(\boldsymbol{U}^2))) - \eta\Gamma_t + \nabla\cdot(h\boldsymbol{U})\Gamma + \nabla(\boldsymbol{U}\cdot(h\Gamma) - \eta\nabla\cdot(h\boldsymbol{U}_t) + \tfrac{1}{2}(\nabla\cdot(h\boldsymbol{U}))^2)], \tag{4.3 b}$$

$$\Lambda_{40}^{\rm III} = [\beta_1 h^4\nabla(\nabla^4\eta) + h^4(\beta_1+\tfrac{1}{3}\alpha_1-\tfrac{1}{45})\nabla(\nabla^2(\nabla\cdot\boldsymbol{U}_t)) + \beta_2 h^3\nabla h\nabla^4\eta + (\beta_2+\tfrac{7}{3}\alpha_1+\tfrac{2}{3}\alpha_2-\tfrac{2}{9})h^3\nabla h\nabla^2(\nabla\cdot\boldsymbol{U}_t)] + O(\mu^2), \tag{4.3 c}$$

$$\begin{aligned}\Lambda_{41}^{\rm III} = {}& [\tfrac{1}{45}h^3\eta\nabla(\nabla^2(\nabla\cdot\boldsymbol{U}_t)) + \alpha_1 h^3\nabla(\nabla\cdot(\nabla\eta\nabla\cdot\boldsymbol{U}_t)) \\ & + (\tfrac{2}{3}\alpha_1-\tfrac{1}{9})h^3\nabla(\nabla\cdot(\eta\nabla(\nabla\cdot\boldsymbol{U}_t))) + (\tfrac{1}{9}-\tfrac{2}{3}\alpha_1)h^4\nabla(\nabla\cdot(\nabla\cdot\boldsymbol{U}(\nabla(\nabla\cdot\boldsymbol{U})))) \\ & - \tfrac{1}{45}h^4\nabla\cdot\boldsymbol{U}\nabla(\nabla^2(\nabla\cdot\boldsymbol{U})) + \tfrac{1}{18}h^4\nabla(\nabla(\nabla\cdot\boldsymbol{U}))^2 - \tfrac{1}{45}h^4\nabla(\boldsymbol{U}\cdot\nabla(\nabla^2(\nabla\cdot\boldsymbol{U}))) \\ & + \tfrac{1}{3}\alpha_1 h^4\nabla(\nabla^2(\boldsymbol{U}\cdot(\nabla(\nabla\cdot\boldsymbol{U})))) + \tfrac{1}{2}\beta_1 h^4\nabla(\nabla^4(\boldsymbol{U}^2))] + O(\mu),\end{aligned} \tag{4.3 d}$$

and where $\Lambda_{22}^{\rm III} = \Lambda_{22}^{\rm II}$ and $\Lambda_{23}^{\rm III} = \Lambda_{23}^{\rm II}$ defined by (3.8 *c*) and (3.8 *d*). The four coefficients α_1, α_2, β_1 and β_2 are yet to be determined. It turns out that the set (α_1, β_1) governs the linear dispersion relation, while the set (α_2, β_2) can be used to optimize the linear shoaling gradient.

(*c*) *Fourier analysis of equations on a horizontal bottom*

The analysis of the dispersion and nonlinearity characteristics of the enhanced higher-order equations (3.2) and (4.2) follows the procedure outlined in § 3 *b* and again we shall look for first-, second- and third-order Fourier solutions defined by (3.10). It is emphasized that this analysis will not involve nonlinear terms with powers of ϵ higher than two although such terms appear in the formulation. In one dimension and on a horizontal bottom the equations reduce to

$$\eta_t + hU_x + \epsilon(\eta U)_x = 0 \tag{4.4 a}$$

and

$$\begin{aligned} & U_t + \eta_x - \mu^2((\alpha_1+\tfrac{1}{3})h^2U_{xxt} + \alpha_1 h^2\eta_{xxx}) \\ & + \mu^4((\beta_1+\tfrac{1}{3}\alpha_1-\tfrac{1}{45})h^4U_{xxxxt} + \beta_1 h^4\eta_{xxxxx}) + \epsilon UU_x \\ & + \epsilon\mu^2(-\tfrac{2}{3}h\eta U_{xxt} - h\eta_x U_{xt} + \tfrac{1}{3}h^2U_xU_{xx} - \tfrac{1}{3}h^2UU_{xxx} - \alpha_1 h^2(UU_x)_{xx}) \\ & + \epsilon\mu^4(\tfrac{1}{45}h^3\eta U_{xxxxt} + \alpha_1 h^3(\eta_x U_{xt})_{xx} + (\tfrac{2}{3}\alpha_1-\tfrac{1}{9})h^3(\eta U_{xxt})_{xx} \\ & + (\tfrac{1}{9}-\tfrac{2}{3}\alpha_1)h^4(U_xU_{xx})_{xx} - \tfrac{1}{45}h^4U_xU_{xxxx} + \tfrac{1}{9}h^4U_{xx}U_{xxx} \\ & - \tfrac{1}{45}h^4(UU_{xxxx})_x + \tfrac{1}{3}\alpha_1 h^4(UU_{xx})_{xxx} + \beta_1 h^4(UU_x)_{xxxx}) \\ & + \epsilon^2\mu^2(\tfrac{1}{6}\eta^2U_{xxt} - \tfrac{1}{3}\eta hU_xU_{xx} - \tfrac{1}{3}hU_{xx}(\eta U)_x + h(\eta U_x^2)_x \\ & - \tfrac{1}{2}(\eta^2U_{xt})_x - \tfrac{2}{3}h(\eta UU_{xx})_x) = O(\epsilon^3\mu^2, \epsilon^2\mu^4, \mu^6). \end{aligned} \tag{4.4 b}$$

(i) *First-order solution*

Inserting (3.10) into (4.4) and collecting terms of $O(\epsilon^0)$ leads to (3.13) with

$$\left.\begin{aligned} m_{11}^{(1)} = \omega, \quad m_{12}^{(1)} = -kh, \quad m_{21}^{(1)} = -k(1+\alpha_1\kappa^2+\beta_1\kappa^4), \\ m_{22}^{(1)} = \omega(1+(\alpha_1+\tfrac{1}{3})\kappa^2+(\beta_1+\tfrac{1}{3}\alpha_1-\tfrac{1}{45})\kappa^4). \end{aligned}\right\} \tag{4.5}$$

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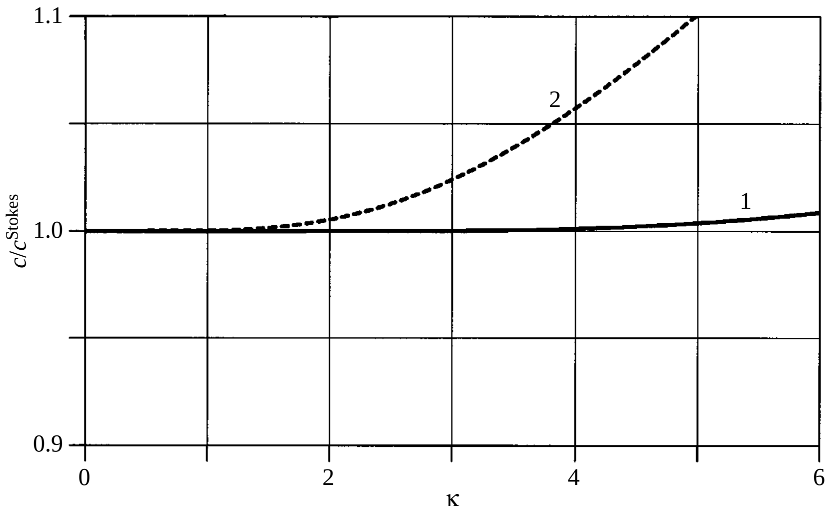

Figure 3. Ratio of phase celerity, c/c^{Stokes}, where c is determined by (4.6) and c^{Stokes} by (3.16). Boussinesq equations include (1) $O(\mu^4)$ with $\alpha_1 = \frac{1}{9}$, $\beta_1 = \frac{1}{945}$, Padé [4,4]; (2) $O(\mu^2)$ with $\alpha_1 = \frac{1}{15}$, Padé [2,2].

Hence, at first order we get (3.15 *a*) for U_1, and the dispersion relation

$$\frac{\omega^2}{k^2 h} = \frac{1 + \alpha_1\kappa^2 + \beta_1\kappa^4}{1 + (\alpha_1 + \frac{1}{3})\kappa^2 + (\beta_1 + \frac{1}{3}\alpha_1 - \frac{1}{45})\kappa^4}. \tag{4.6}$$

Equation (4.6) has the form of the Padé [4,4] expansion given by (3.17), which is matched identically by choosing the parameters $\alpha_1 = \frac{1}{9}$ and $\beta_1 = \frac{1}{945}$. Figure 3 shows the ratio between (4.6) and the target (3.16) as a function of κ. The accuracy is seen to be excellent even for κ as large as 6, which is twice the traditional deep-water limit. In comparison with the characteristics of the original higher-order equations (shown in figure 1), we have improved the accuracy considerably and by doing so removed the singularity at $\kappa \approx 4.2$.

If $\beta_1 = 0$ and $\alpha_1 = \frac{1}{15}$, then the linear μ^4 terms vanish from (4.4 *b*), and (4.6) reduces to Padé [2,2] dispersion. This value of α_1 was found by Madsen *et al.* (1991) using the enhancement technique on $O(\mu^2)$ equations, i.e. without actually deriving the $O(\mu^4)$ terms. This shows the equivalence between the two alternative enhancement procedures described in § 4*a*. On a sloping bottom a similar principle applies, see the discussion after (4.11 *b*) in § 4 *d*. For reference this case is included in figure 3.

(ii) *Second-order solution*

Continuing the analysis to second order and collecting terms of $O(\epsilon)$ leads to (3.18) with

$$\left.\begin{aligned} m_{11}^{(2)} = 2\omega, \quad m_{12}^{(2)} = -2kh, \quad m_{21}^{(2)} = -2k(1 + 4\alpha_1\kappa^2 + 16\beta_1\kappa^4), \\ m_{22}^{(2)} = 2\omega(1 + 4(\alpha_1 + \tfrac{1}{3})\kappa^2 + 16(\beta_1 + \tfrac{1}{3}\alpha_1 - \tfrac{1}{45})\kappa^4), \end{aligned}\right\} \tag{4.7 a}$$

$$F_1 = \omega, \quad F_2 = (\omega^2/2kh)(1 + (4\alpha_1 - \tfrac{5}{3})\kappa^2 + (16\beta_1 - \tfrac{20}{3}\alpha_1 + \tfrac{41}{45})\kappa^4). \tag{4.7 b}$$

Table 2. *Expansion coefficients for the second harmonic*

	Stokes	$(\mu^4, \epsilon\mu^4)$	$(\mu^4, \epsilon\mu^2)$	$(\mu^2, \epsilon\mu^2)$	(μ^2, ϵ)
c_2	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{17}{15}$
c_4	$\frac{7}{45}$	$\frac{7}{45}$	$\frac{88}{945}$	$\frac{1}{25}$	

The general form of the second-order solution is defined by (3.20) with (4.7), while Stokes's reference solution is given by (3.21). It turns out that when using the enhancement coefficients $\alpha_1 = \frac{1}{9}$ and $\beta_1 = \frac{1}{945}$ while retaining the $(\mu^4, \epsilon\mu^4)$ terms in (4.4), the resulting expression for a_2 is free from the singularity which was found in § 3 *b*. An expansion from $\kappa = 0$ of a_2 gives

$$a_2 = \frac{3}{4}\frac{a_1^2}{h}\frac{1}{\kappa^2}(1 + c_2\kappa^2 + c_4\kappa^4 + O(\kappa^6)),$$

where $c_2 = \frac{2}{3}$ and $c_4 = \frac{1}{45}(13 - 63\alpha_1 + 945\beta_1)$, when $(\mu^4, \epsilon\mu^4)$ terms are retained. By using $\alpha_1 = \frac{1}{9}$ and $\beta_1 = \frac{1}{945}$ as determined above, we get $c_4 = \frac{7}{45}$, which is identical to the target value in (3.22). If we retain only $(\mu^4, \epsilon\mu^2)$ terms, which would be a consequence of assuming $\epsilon = O(\mu^2)$, it leads to the neglect of the κ^4 term in (4.7 *b*) and results in $c_2 = \frac{2}{3}$ and

$$c_4 = \tfrac{1}{135}(-2 + 111\alpha_1 + 2115\beta_1).$$

Hence, in this case the chosen set of (α_1, β_1) yields $c_4 = \frac{88}{945}$, which is different from the target.

If we neglect all μ^4 terms and retain only $(\mu^2, \epsilon\mu^2)$ terms, we get $c_2 = \frac{1}{3}(1 + 15\alpha_1)$. Hence with the value of $\alpha_1 = \frac{1}{15}$ (corresponding to a Padé [2,2]), this leads to $c_2 = \frac{2}{3}$, which is in agreement with the target. If we retain only (μ^2, ϵ) terms (i.e. ignoring κ^2 in (4.7 *b*)), we get $c_2 = \frac{1}{9}(8 + 33\alpha_1)$ leading to $c_2 = \frac{17}{15}$, which results in a clear overestimate of a_2. The results are summarized in table 2.

Returning to the discussion from § 3 *b*, it is actually remarkable that we are able to obtain the correct c_4 coefficient with the new enhanced $O(\mu^4, \epsilon\mu^4)$ formulation, while the correct c_2 coefficient can be obtained with the enhanced $O(\mu^2, \epsilon\mu^2)$ formulation. As seen from table 1 this would normally require $O(\mu^6, \epsilon\mu^4)$ and $O(\mu^4, \epsilon\mu^2)$ formulations, respectively. This indicates that the introduced enhancement effectively increases the accuracy of the linear terms with at least a factor of μ^2. Furthermore, it shows that the potential of this linear improvement is fully utilized with regard to the second-order transfer only if a sufficient order of dispersion is retained also in the nonlinear terms.

Figure 4 shows the ratio between a_2 and the target (3.21) as a function of κ. The result obtained by the $(\mu^4, \epsilon\mu^4)$ formulation is quite attractive: for $\kappa < 1$ the second harmonic is almost perfect, while it is gradually underestimated for larger values of κ reaching 50% for $\kappa = 6$. It is also clear from figure 4 that the enhanced $(\mu^2, \epsilon\mu^2)$ formulation is superior in comparison with the enhanced (μ^2, ϵ) formulation, as already indicated by the above analysis.

(iii) *Third-order solution*

Continuing the analysis of (4.4) to third order, we insert (3.10) in (4.4) and collect terms of order $O(\epsilon^2)$. As in § 3 *b* we remove secular $\sin\theta$ terms by introducing (3.24),

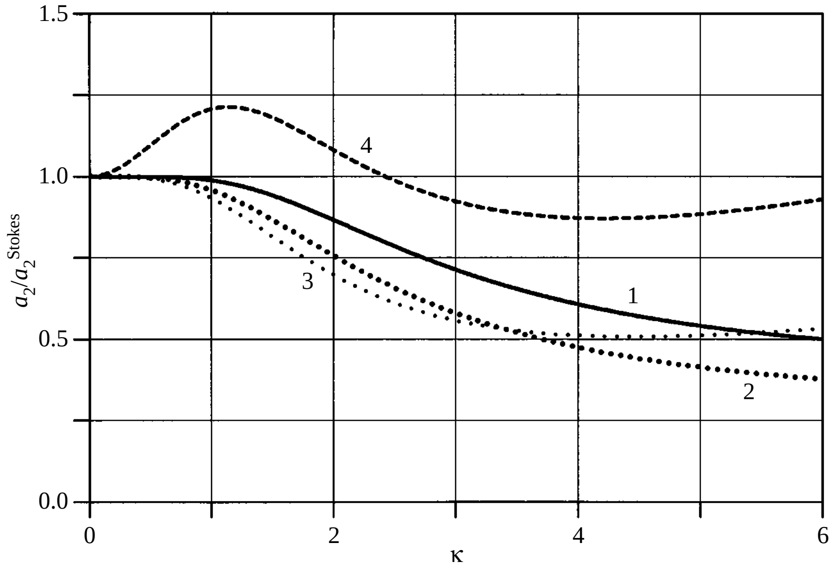

Figure 4. Ratio of second harmonic, a_2/a_2^{Stokes}, where a_2 is determined by (3.20 *a*) combined with (4.7) and a_2^{Stokes} by (3.21). Boussinesq equations include (1) $O(\mu^4, \epsilon\mu^4)$ with $\alpha_1 = \frac{1}{9}$, $\beta_1 = \frac{1}{945}$; (2) $O(\mu^4, \epsilon\mu^2)$ with $\alpha_1 = \frac{1}{9}$, $\beta_1 = \frac{1}{945}$; (3) $O(\mu^2, \epsilon\mu^2)$ with $\alpha_1 = \frac{1}{15}$; (4) $O(\mu^2, \epsilon)$ with $\alpha_1 = \frac{1}{15}$.

which defines the amplitude dispersion (ω_{13}) and a third-order correction to the first-order velocity. Again we focus on the quantities a_3 and ω_{13}. It turns out that when using the enhancement coefficients $\alpha_1 = \frac{1}{9}$ and $\beta_1 = \frac{1}{945}$ while retaining the $(\mu^4, \epsilon\mu^4, \epsilon^2\mu^2)$ terms in (4.4), the resulting expressions for a_3 and ω_{13} are free from the two singularities which were found in § 3 *b*. The ratios between the Boussinesq solutions and the target solutions of (3.25) are shown in figure 5. The nonlinearity is generally significantly underestimated for large values for κ.

(*d*) *Linear shoaling analysis*

Madsen & Sørensen (1992) introduced the linear shoaling gradient as another important quantity to measure the applicability of Boussinesq equations. This quantity is defined by

$$\frac{A_x}{A} = -\frac{h_x}{h}\gamma_0, \tag{4.8}$$

where A is the local wave amplitude, h is the depth, and the shoaling gradient γ_0 is a function of the local wavenumber κ. By combining Stokes's linear theory with conservation of energy flux, Madsen & Sørensen derived the reference gradient,

$$\gamma_0^{\text{Stokes}} = \frac{2\kappa \sinh 2\kappa + 2\kappa^2(1 - \cosh 2\kappa)}{(2\kappa + \sinh 2\kappa)^2}, \tag{4.9}$$

of which a Taylor expansion from $\kappa = 0$ yields

$$\gamma_0^{\text{Stokes}} = \tfrac{1}{4} - \tfrac{1}{4}\kappa^2 + \tfrac{1}{18}\kappa^4 + \tfrac{1}{540}\kappa^6 - \tfrac{11}{3150}\kappa^8 + O(\kappa^{10}). \tag{4.10}$$

Recently, Chen & Liu (1995) argued that the gradient is not a good quantity for measuring the linear shoaling effect, because a deviation from the target gradient has

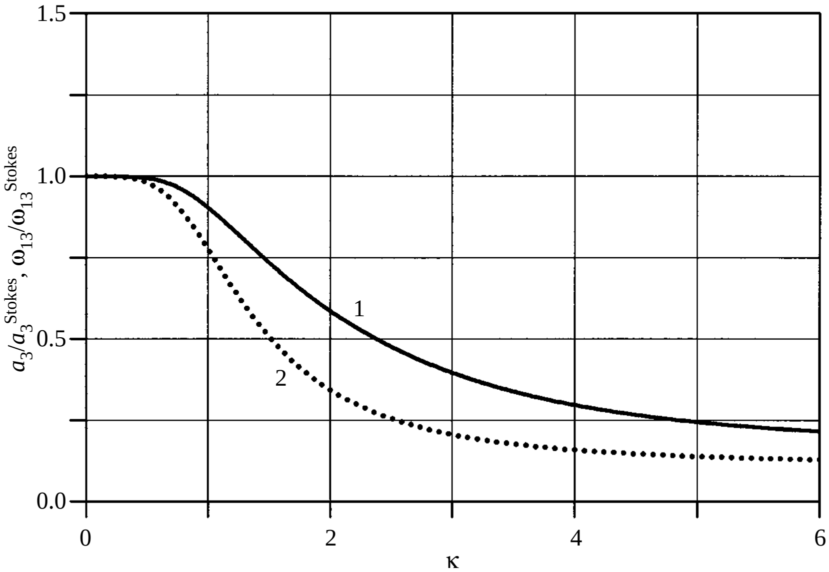

Figure 5. Ratios of (1) a_3/a_3^{Stokes} and (2) $\omega_{13}/\omega_{13}^{\text{Stokes}}$ for enhanced Boussinesq equations include $O(\mu^4, \epsilon\mu^4, \epsilon^2\mu^2)$ with $\alpha_1 = \frac{1}{9}$, $\beta_1 = \frac{1}{945}$.

less effect as the relative depth increases and that errors occurring in intermediate depths and deep water exaggerates the actual difference of the resulting shoaling amplitude. Although their argument is easily appreciated, there is really no excuse for accepting a discrepancy from the target gradient, considering how little it takes to obtain a high accuracy. Furthermore, the shoaling gradient is by far the most obvious analytical measure of the linear shoaling characteristics of the mass and momentum equations.

In this section we shall derive the shoaling gradient corresponding to the new enhanced higher-order Boussinesq equations, and determine the remaining two free coefficients α_2 and β_2 to optimize the linear shoaling characteristics. In the analysis we shall assume a mildly sloping bottom, thus retaining only first derivatives of the depth and neglecting all nonlinear terms. Hence, in one dimension, (4.2) and (3.2) simplify to

$$\eta_t + hU_x + h_x U = O(\epsilon) \tag{4.11 a}$$

and

$$\begin{aligned} U_t + \eta_x - \mu^2((\alpha_1 + \tfrac{1}{3})h^2 U_{xxt} + \alpha_1 h^2 \eta_{xxx}) - \mu^2 h_x((1 + 2\alpha_2)hU_{xt} + 2\alpha_2 h\eta_{xx}) \\ + \mu^4((\beta_1 + \tfrac{1}{3}\alpha_1 - \tfrac{1}{45})h^4 U_{xxxxt} + \beta_1 h^4 \eta_{xxxxx}) \\ + \mu^4 h_x((\beta_2 + \tfrac{7}{3}\alpha_1 + \tfrac{2}{3}\alpha_2 - \tfrac{2}{9})h^3 U_{xxxt} + \beta_2 h^3 \eta_{xxxx}) = O(\epsilon). \end{aligned} \tag{4.11 b}$$

If $(\alpha_1, \alpha_2, \beta_1, \beta_2) = (\frac{1}{15}, -\frac{1}{10}, 0, 0)$, then all $O(\mu^4)$ terms vanish. This set of parameters (α_1, α_2) was obtained by Schäffer & Madsen (1995*a*) using the enhancement technique on $O(\mu^2)$ equations, i.e. without actually deriving the $O(\mu^4)$ terms. While α_1 was determined by matching with a target dispersion relation, α_2 was obtained by matching the embedded shoaling gradient with (4.10) to the order μ^4. Again, this shows the equivalence between the two alternative enhancement procedures using

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(a) the method of extinguishing high-order terms or using (b) a match to dispersion and shoaling as known from fully dispersive theory.

Following the procedure of Schäffer & Madsen (1995*a*), we look for solutions of the form

$$\eta(x,t) = A(x)\mathrm{e}^{\mathrm{i}(\omega t-\Psi(x))}, \qquad (4.12\,a)$$

$$U(x,t) = D(x)(1+\mathrm{i}\sigma(x)h_x)\mathrm{e}^{\mathrm{i}(\omega t-\Psi(x))}, \qquad (4.12\,b)$$

where

$$\frac{\mathrm{d}\Psi}{\mathrm{d}x} = k(x) \qquad (4.13)$$

and where A, D and σ are real slowly varying functions of x. On a constant depth U is in phase with η, while for small bottom slopes a small phase shift should be permitted. This is the reason for introducing the σ term in (4.12 *b*). First derivatives of A, D, σ, k and h are assumed to be small, and thus products of derivatives as well as higher derivatives of these quantities are neglected in the analysis.

Substituting (4.12) into the continuity equation (4.11 *a*) and collecting real and imaginary terms leads to two equations: one which is free from derivatives of the slow variables,

$$D = \omega A/kh, \qquad (4.14)$$

and one which contains terms proportional to first derivatives of the slow variables,

$$\frac{D_x}{D} + (1+\sigma kh)\frac{h_x}{h} = 0. \qquad (4.15)$$

Differentiating (4.14) with respect to x leads to

$$\frac{D_x}{D} = \frac{A_x}{A} - \frac{k_x}{k} - \frac{h_x}{h}, \qquad (4.16\,a)$$

which in combination with (4.15) yields

$$\sigma kh\frac{h_x}{h} = \frac{k_x}{k} - \frac{A_x}{A}. \qquad (4.16\,b)$$

The next step is to substitute (4.12) into the momentum equation (4.11 *b*). The imaginary part in combination with (4.14) leads to the dispersion relation (4.6), while the real part contains terms proportional to first derivatives of the slow variables. By the use of (4.14) and (4.16), this equation can be expressed as

$$\gamma_1\frac{A_x}{A} + \gamma_2\frac{k_x}{k} + \gamma_3\frac{h_x}{h} = 0, \qquad (4.17)$$

where

$$\gamma_1 \equiv 1+\zeta^2+\kappa^2(3\alpha_1-\zeta^2(\alpha_1+\tfrac{1}{3}))+\kappa^4(5\beta_1-3\zeta^2(\beta_1+\tfrac{1}{3}\alpha_1-\tfrac{1}{45})), \qquad (4.18\,a)$$

$$\gamma_2 \equiv -\zeta^2+3\alpha_1\kappa^2+\kappa^4(10\beta_1-3\zeta^2(\beta_1+\tfrac{1}{3}\alpha_1-\tfrac{1}{45})), \qquad (4.18\,b)$$

$$\gamma_3 \equiv \kappa^2(2\alpha_2+\zeta^2(2\alpha_1-2\alpha_2-\tfrac{1}{3}))+\kappa^4(\beta_2+\zeta^2(4\beta_1-\beta_2-\alpha_1-\tfrac{2}{3}\alpha_2+\tfrac{2}{15})), \qquad (4.18\,c)$$

and where ζ is defined by

$$\zeta \equiv \frac{\omega}{k\sqrt{h}} = \frac{c'}{\sqrt{gh'}}. \qquad (4.19)$$

The first derivative of the wavenumber k can be expressed in terms of the first derivative of h by differentiating the dispersion relation (4.6) with respect to x, and this leads to

$$\gamma_4 \frac{k_x}{k} + \gamma_5 \frac{h_x}{h} = 0, \tag{4.20}$$

where

$$\gamma_4 \equiv 2[1 + \kappa^2(2\alpha_1 - \zeta^2(\alpha_1 + \tfrac{1}{3})) + \kappa^4(3\beta_1 - 2\zeta^2(\beta_1 + \tfrac{1}{3}\alpha_1 - \tfrac{1}{45}))], \tag{4.21 a}$$

$$\gamma_5 \equiv 1 + \kappa^2(3\alpha_1 - 2\zeta^2(\alpha_1 + \tfrac{1}{3})) + \kappa^4(5\beta_1 - 4\zeta^2(\beta_1 + \tfrac{1}{3}\alpha_1 - \tfrac{1}{45})). \tag{4.21 b}$$

Finally, by substituting (4.20) into (4.17), we get an expression for the linear shoaling gradient:

$$\frac{A_x}{A} = -\frac{h_x}{h}\gamma_0, \quad \gamma_0 \equiv \frac{\gamma_3\gamma_4 - \gamma_2\gamma_5}{\gamma_1\gamma_4}. \tag{4.22}$$

For $(\alpha_1, \beta_1) = (\frac{1}{9}, \frac{1}{945})$ the expansion of (4.22) from $\kappa = 0$ gives

$$\gamma_0 = \tfrac{1}{4} - \tfrac{1}{4}\kappa^2 + \tfrac{1}{18}\kappa^4 + \tfrac{1}{540}(\tfrac{43}{21} - 12\alpha_2 + 90\beta_2)\kappa^6 + \tfrac{1}{3150}(-\tfrac{1006}{81} + \tfrac{130}{9}\alpha_2 - \tfrac{280}{3}\beta_2)\kappa^8 + O(\kappa^{10}). \tag{4.23}$$

Matching (4.23) with (4.10) to $O(\kappa^8)$ gives $(\alpha_2, \beta_2) = (\frac{1}{6}, \frac{2}{189})$. However, since terms of much higher order than $O(\kappa^8)$ are inevitable in (4.22), we prefer to follow the procedure by Schäffer & Madsen (1995*a*) and determine the coefficients (α_2, β_2) from minimization of the integral error,

$$\frac{1}{\kappa_0}\int_0^{\kappa_0} (\gamma_0^{\text{Stokes}} - \gamma_0)^2 \, d\kappa.$$

With $\kappa_0 = 6$ this procedure determines the set $(\alpha_2, \beta_2) = (0.146\,488, 0.007\,983\,59)$ with a minimum integral error of 3.5×10^{-6}.

A comparison between (4.22) and (4.9) is shown in figure 6 for both sets of coefficients and obviously the set determined from the principle of minimization is by far the most attractive. With this set the agreement is seen to be excellent for $0 \leqslant \kappa \leqslant 6$. Note that $\kappa = 6$ corresponds to twice the traditional deep water limit.

5. Equations in terms of the velocity at an arbitrary z location, $\tilde{u}$

(*a*) *Derivation of* $(\mu^4, \epsilon^5\mu^4)$ *equations*

As an alternative to the traditional $\boldsymbol{U}$ formulation, Nwogu (1993) introduced the velocity vector at an arbitrary z location and derived a set of lower-order Boussinesq equations retaining terms of order $O(\epsilon)$ and $O(\mu^2)$. With a specific choice of this z location, Nwogu achieved the same quality of dispersion (Padé [2,2]) as previously obtained by Madsen *et al.* (1991) on the basis of the enhanced lower-order $\boldsymbol{U}$ formulation. It is therefore of interest to pursue a higher-order formulation of the equations of Nwogu. Recently, Wei *et al.* (1995) extended these equations to include terms of order $O(\mu^2, \epsilon^3\mu^2)$. In this section we shall derive and analyse equations of order $O(\mu^4, \epsilon^5\mu^4)$ by reformulating the higher-order Boussinesq-type equations (2.15)

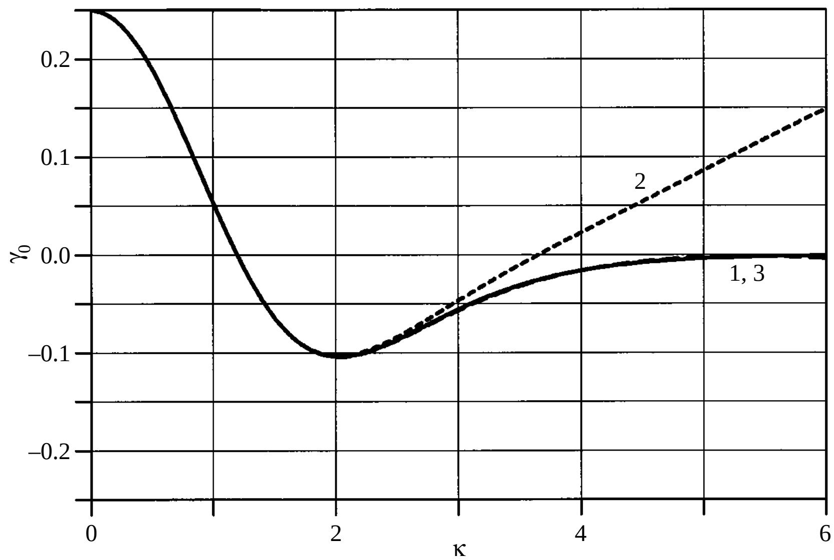

Figure 6. Linear shoaling gradient, γ_0 defined by the relation $A_x/A = -\gamma_0 h_x/h$. (1) Equation (4.9), Stokes's first order (full line); (2) equation (4.22) with $(\alpha_2, \beta_2) = (\frac{1}{6}, \frac{2}{189})$; (3) equation (4.22) with $(\alpha_2, \beta_2) = (0.1465, 0.007\,98)$.

and (2.19) in terms of the variable $\tilde{\boldsymbol{u}}$ denoting the velocity vector at $z = \tilde{z}(x, y)$. The first step is to insert (2.14) in (2.8 *a*) and after truncating at μ^6 we obtain

$$\boldsymbol{u}(x,y,z,t) = \hat{\boldsymbol{u}} - \mu^2(z\nabla(\nabla\cdot(h\hat{\boldsymbol{u}})) + \tfrac{1}{2}z^2\nabla(\nabla\cdot\hat{\boldsymbol{u}})) + \mu^4(\tfrac{1}{24}z^4\nabla(\nabla^2(\nabla\cdot\hat{\boldsymbol{u}})) + \tfrac{1}{6}z^3\nabla(\nabla^2(\nabla\cdot(h\hat{\boldsymbol{u}}))) + z\nabla(\nabla\cdot(h^2\hat{\Gamma}))) + O(\mu^6), \quad (5.1)$$

where $\hat{\Gamma}$ is defined by (2.17). In order to replace $\hat{\boldsymbol{u}}$ by $\tilde{\boldsymbol{u}}$ in the mass and momentum equations, we establish the inverse relation of (5.1) (at $z = \tilde{z}(x, y)$) in which $\hat{\boldsymbol{u}}$ is expressed in terms of $\tilde{\boldsymbol{u}}$. As in § 3 *a*, this is achieved by the use of successive substitutions starting at lowest order in μ^2 and results in

$$\begin{aligned}\hat{\boldsymbol{u}} = \tilde{\boldsymbol{u}} + \mu^2\tilde{\Gamma} + \mu^4[&-\tfrac{1}{24}\tilde{z}^4\nabla(\nabla^2(\nabla\cdot\tilde{\boldsymbol{u}})) - \tfrac{1}{6}\tilde{z}^3\nabla(\nabla^2(\nabla\cdot(h\tilde{\boldsymbol{u}})))\\ &+ \tilde{z}\nabla(\nabla\cdot((h\tilde{z}+\tfrac{1}{2}h^2)\nabla(\nabla\cdot(h\tilde{\boldsymbol{u}})) + (\tfrac{1}{2}h\tilde{z}^2 - \tfrac{1}{6}h^3)\nabla(\nabla\cdot\tilde{\boldsymbol{u}}))\\ &+ \tfrac{1}{2}\tilde{z}^2\nabla(\nabla\cdot\tilde{\Gamma})] + O(\mu^6),\end{aligned} \quad (5.2\,a)$$

where

$$\tilde{\Gamma} \equiv \tilde{z}\nabla(\nabla\cdot(h\tilde{\boldsymbol{u}})) + \tfrac{1}{2}\tilde{z}^2\nabla(\nabla\cdot\tilde{\boldsymbol{u}}). \quad (5.2\,b)$$

By substituting (5.2) in (2.19) we get

$$\begin{aligned}\boldsymbol{Q} = \tilde{\boldsymbol{u}}(h+\epsilon\eta) &+ \mu^2[(\tfrac{1}{2}h^2 + h\tilde{z} + \epsilon\eta\tilde{z} - \tfrac{1}{2}\epsilon^2\eta^2)\nabla(\nabla\cdot(h\tilde{\boldsymbol{u}}))\\ &+ (\tfrac{1}{2}h\tilde{z}^2 - \tfrac{1}{6}h^3 + \tfrac{1}{2}\epsilon\eta\tilde{z}^2 - \tfrac{1}{6}\epsilon^3\eta^3)\nabla(\nabla\cdot\tilde{\boldsymbol{u}})]\\ &+ \mu^4[(\tfrac{1}{120}h^5 - \tfrac{1}{24}h\tilde{z}^4 - \tfrac{1}{24}\epsilon\eta\tilde{z}^4 + \tfrac{1}{120}\epsilon^5\eta^5)\nabla(\nabla^2(\nabla\cdot\tilde{\boldsymbol{u}}))\\ &- (\tfrac{1}{24}h^4 + \tfrac{1}{6}h\tilde{z}^3 + \tfrac{1}{6}\epsilon\eta\tilde{z}^3 - \tfrac{1}{24}\epsilon^4\eta^4)\nabla(\nabla^2(\nabla\cdot(h\tilde{\boldsymbol{u}})))\\ &+ (\tfrac{1}{2}h^2 + h\tilde{z} + \epsilon\eta\tilde{z} - \tfrac{1}{2}\epsilon^2\eta^2)\nabla(\nabla\cdot((h\tilde{z}+\tfrac{1}{2}h^2)\nabla(\nabla\cdot(h\tilde{\boldsymbol{u}}))\\ &+ (\tfrac{1}{2}h\tilde{z}^2 - \tfrac{1}{6}h^3)\nabla(\nabla\cdot\tilde{\boldsymbol{u}})) + (\tfrac{1}{2}h\tilde{z}^2 - \tfrac{1}{6}h^3 + \tfrac{1}{2}\epsilon\eta\tilde{z}^2 - \tfrac{1}{6}\epsilon^3\eta^3)\nabla(\nabla\cdot\tilde{\Gamma})] + O(\mu^6),\end{aligned} \quad (5.3)$$

which in combination with (2.18) defines the continuity equation expressed in terms of $\tilde{\boldsymbol{u}}$.

The next step is to substitute (5.2) into (2.15) and (2.16) and after collecting terms in powers of ϵ and μ, we determine the following momentum equation in terms of $\tilde{\boldsymbol{u}}$:

$$\tilde{\boldsymbol{u}}_t + \nabla\eta + \tfrac{1}{2}\epsilon\nabla(\tilde{\boldsymbol{u}})^2 + \mu^2[\Lambda_{20}^{\mathrm{IV}} + \epsilon\Lambda_{21}^{\mathrm{IV}} + \epsilon^2\Lambda_{22}^{\mathrm{IV}} + \epsilon^3\Lambda_{23}^{\mathrm{IV}}] + \mu^4[\Lambda_{40}^{\mathrm{IV}} + \epsilon\Lambda_{41}^{\mathrm{IV}} + \epsilon^2\Lambda_{42}^{\mathrm{IV}} + \epsilon^3\Lambda_{43}^{\mathrm{IV}} + \epsilon^4\Lambda_{44}^{\mathrm{IV}} + \epsilon^5\Lambda_{45}^{\mathrm{IV}}] = O(\mu^6), \quad (5.4)$$

where

$$\Lambda_{20}^{\mathrm{IV}} = \tilde{\Gamma}_t, \quad (5.5\,a)$$

$$\Lambda_{21}^{\mathrm{IV}} = \nabla[\tilde{\boldsymbol{u}}\cdot\tilde{\Gamma} - \eta\nabla\cdot(h\tilde{\boldsymbol{u}}_t) + \tfrac{1}{2}(\nabla\cdot(h\tilde{\boldsymbol{u}}))^2], \quad (5.5\,b)$$

$$\Lambda_{22}^{\mathrm{IV}} = \nabla[-\tfrac{1}{2}\eta^2\nabla\cdot\tilde{\boldsymbol{u}}_t - \eta\tilde{\boldsymbol{u}}\cdot\nabla(\nabla\cdot(h\tilde{\boldsymbol{u}})) + \eta(\nabla\cdot\tilde{\boldsymbol{u}})(\nabla\cdot(h\tilde{\boldsymbol{u}}))], \quad (5.5\,c)$$

$$\Lambda_{23}^{\mathrm{IV}} = \nabla[-\tfrac{1}{2}\eta^2\tilde{\boldsymbol{u}}\cdot\nabla(\nabla\cdot\tilde{\boldsymbol{u}}) + \tfrac{1}{2}\eta^2(\nabla\cdot\tilde{\boldsymbol{u}})^2], \quad (5.5\,d)$$

$$\Lambda_{40}^{\mathrm{IV}} = -\tfrac{1}{24}\tilde{z}^4\nabla(\nabla^2(\nabla\cdot\tilde{\boldsymbol{u}}_t)) - \tfrac{1}{6}\tilde{z}^3\nabla(\nabla^2(\nabla\cdot(h\tilde{\boldsymbol{u}}_t))) + \tilde{z}\nabla(\nabla\cdot\bar{\Gamma}_t) + \tfrac{1}{2}\tilde{z}^2\nabla(\nabla\cdot\tilde{\Gamma}_t), \quad (5.5\,e)$$

$$\Lambda_{41}^{\mathrm{IV}} = \nabla[\tfrac{1}{2}\tilde{\Gamma}^2 + \tilde{\boldsymbol{u}}\cdot(-\tfrac{1}{24}\tilde{z}^4\nabla(\nabla^2(\nabla\cdot\tilde{\boldsymbol{u}})) - \tfrac{1}{6}\tilde{z}^3\nabla(\nabla^2(\nabla\cdot(h\tilde{\boldsymbol{u}}))) + \tilde{z}\nabla(\nabla\cdot\bar{\Gamma}) + \tfrac{1}{2}\tilde{z}^2\nabla(\nabla\cdot\tilde{\Gamma})) - \eta\nabla\cdot\bar{\Gamma}_t + \nabla\cdot(h\tilde{\boldsymbol{u}})\nabla\cdot\bar{\Gamma}], \quad (5.5\,f)$$

$$\Lambda_{42}^{\mathrm{IV}} = \nabla[-\tfrac{1}{2}\eta^2\nabla\cdot\tilde{\Gamma}_t - \eta\nabla(\nabla\cdot(h\tilde{\boldsymbol{u}}))\cdot\tilde{\Gamma} + \eta(\nabla\cdot(h\tilde{\boldsymbol{u}}))(\nabla\cdot\tilde{\Gamma}) - \eta\tilde{\boldsymbol{u}}\cdot\nabla(\nabla\cdot\bar{\Gamma}) + \eta(\nabla\cdot\tilde{\boldsymbol{u}})(\nabla\cdot\bar{\Gamma})], \quad (5.5\,g)$$

$$\Lambda_{43}^{\mathrm{IV}} = \nabla[-\tfrac{1}{2}\eta^2\tilde{\boldsymbol{u}}\cdot\nabla(\nabla\cdot\tilde{\Gamma}) - \tfrac{1}{2}\eta^2(\nabla(\nabla\cdot\tilde{\boldsymbol{u}}))\cdot\tilde{\Gamma} + \eta^2(\nabla\cdot\tilde{\boldsymbol{u}})(\nabla\cdot\tilde{\Gamma}) + \tfrac{1}{6}\eta^3\nabla^2(\nabla\cdot(h\tilde{\boldsymbol{u}}_t)) + \tfrac{1}{2}\eta^2(\nabla(\nabla\cdot(h\tilde{\boldsymbol{u}})))^2 - \tfrac{1}{2}\eta^2\nabla\cdot(h\tilde{\boldsymbol{u}})\nabla^2(\nabla\cdot(h\tilde{\boldsymbol{u}}))], \quad (5.5\,h)$$

$$\Lambda_{44}^{\mathrm{IV}} = \nabla[\tfrac{1}{2}\eta^3\nabla(\nabla\cdot\tilde{\boldsymbol{u}})\cdot\nabla(\nabla\cdot(h\tilde{\boldsymbol{u}})) - \tfrac{1}{2}\eta^3(\nabla\cdot\tilde{\boldsymbol{u}})\nabla^2(\nabla\cdot(h\tilde{\boldsymbol{u}})) + \tfrac{1}{6}\eta^3\tilde{\boldsymbol{u}}\cdot\nabla(\nabla^2(\nabla\cdot(h\tilde{\boldsymbol{u}}))) - \tfrac{1}{6}\eta^3\nabla\cdot(h\tilde{\boldsymbol{u}})\nabla^2(\nabla\cdot\tilde{\boldsymbol{u}}) + \tfrac{1}{24}\eta^4\nabla^2(\nabla\cdot\tilde{\boldsymbol{u}}_t)], \quad (5.5\,i)$$

$$\Lambda_{45}^{\mathrm{IV}} = \nabla[\tfrac{1}{8}\eta^4(\nabla(\nabla\cdot\tilde{\boldsymbol{u}}))^2 + \tfrac{1}{24}\eta^4\tilde{\boldsymbol{u}}\cdot\nabla(\nabla^2(\nabla\cdot\tilde{\boldsymbol{u}})) - \tfrac{1}{6}\eta^4(\nabla\cdot\tilde{\boldsymbol{u}})\nabla^2(\nabla\cdot\tilde{\boldsymbol{u}})], \quad (5.5\,j)$$

and where $\tilde{\Gamma}$ is given by (5.2 b), while $\bar{\Gamma}$ is given by

$$\bar{\Gamma} \equiv (h\tilde{z} + \tfrac{1}{2}h^2)\nabla(\nabla\cdot(h\tilde{\boldsymbol{u}}) + (\tfrac{1}{2}h\tilde{z}^2 - \tfrac{1}{6}h^3)\nabla(\nabla\cdot\tilde{\boldsymbol{u}}). \quad (5.5\,k)$$

Notice that at this stage no relation between ϵ and μ has been assumed, and that the combination of (5.3) and (5.4) represents a complete system of higher-order Boussinesq-type equations truncated at $O(\mu^6)$ and retaining all nonlinear terms of order $O(\mu^4)$. If terms of order $O(\mu^4)$ are neglected, the equations reduce to the equations of Wei *et al.* (1995), while a further neglect of terms of $O(\epsilon\mu^2)$ leads to the equations of Nwogu (1993).

(b) *Fourier analysis of equations on a horizontal bottom*

The analysis of the dispersion and nonlinearity characteristics of the μ^4 equations in terms of $\tilde{\boldsymbol{u}}$ will follow the procedure outlined in § 3 b. Again we perform the Fourier analysis under the assumption of $\epsilon \ll 1$ (weakly nonlinear solutions) and arbitrary μ, although the derivation of the equations have been formally based on $\mu \ll 1$ and arbitrary ϵ. In the analysis we look for first-, second- and third-order solutions of the form,

$$\left.\begin{aligned} \eta &= a_1\cos\theta + \epsilon a_2\cos 2\theta + \epsilon^2 a_3\cos 3\theta,\\ \tilde{\boldsymbol{u}} &= \tilde{u}_1\cos\theta + \epsilon\tilde{u}_2\cos 2\theta + \epsilon^2\tilde{u}_3\cos 3\theta,\end{aligned}\right\} \tag{5.6}$$

where $\theta = \omega t - kx$, and it is emphasized that the analysis will not involve nonlinear terms with powers of ϵ higher than 2, although higher degrees of nonlinearity appear in the complete formulation. In one dimension and on a horizontal bottom, (5.3) and (5.4) reduce to

$$\begin{aligned} &\eta_t + h\tilde{u}_x + \mu^2(\alpha+\tfrac{1}{3})h^3\tilde{u}_{xxx} + \mu^4\tfrac{5}{6}(\alpha+\tfrac{2}{5})^2h^5\tilde{u}_{xxxxx}\\ &\quad+\epsilon\frac{\partial}{\partial x}\{\eta\tilde{u} + \mu^2(\alpha h^2\eta\tilde{u}_{xx}) + \mu^4(\tfrac{5}{6}\alpha(\alpha+\tfrac{2}{5})h^4\eta\tilde{u}_{xxxx})\}\\ &\quad-\epsilon^2\frac{\partial}{\partial x}\{\tfrac{1}{2}\mu^2h\eta^2\tilde{u}_{xx} + \tfrac{1}{2}\mu^4(\alpha+\tfrac{1}{3})h^3\eta^2\tilde{u}_{xxxx}\} = O(\epsilon^3\mu^2, \epsilon^3\mu^4, \mu^6)\end{aligned} \tag{5.7 a}$$

and

$$\begin{aligned} &\tilde{u}_t + \eta_x + \mu^2(\alpha h^2\tilde{u}_{xxt}) + \mu^4(\tfrac{5}{6}\alpha(\alpha+\tfrac{2}{5})h^4\tilde{u}_{xxxxt}) + \epsilon\tilde{u}\tilde{u}_x\\ &\quad+\epsilon\frac{\partial}{\partial x}\{\mu^2(\alpha h^2\tilde{u}\tilde{u}_{xx} + \tfrac{1}{2}h^2(\tilde{u}_x)^2 - h\eta\tilde{u}_{xt}) + \mu^4(\tfrac{5}{6}\alpha(\alpha+\tfrac{2}{5})h^4\tilde{u}\tilde{u}_{xxxx}\\ &\quad+(\alpha+\tfrac{1}{3})h^4\tilde{u}_x\tilde{u}_{xxx} + \tfrac{1}{2}\alpha^2h^4(\tilde{u}_{xx})^2 - (\alpha+\tfrac{1}{3})h^3\eta\tilde{u}_{xxxt})\}\\ &\quad+\epsilon^2\frac{\partial}{\partial x}\{\mu^2(-\tfrac{1}{2}\eta^2\tilde{u}_{xt} + h\eta(\tilde{u}_x^2 - \tilde{u}\tilde{u}_{xx})) + \mu^4(\alpha h^3\eta(\tilde{u}_x\tilde{u}_{xxx} - \tilde{u}_{xx}^2)\\ &\quad-\tfrac{1}{2}\alpha h^2\eta^2\tilde{u}_{xxxt} + (\alpha+\tfrac{1}{3})h^3\eta(\tilde{u}_x\tilde{u}_{xxx} - \tilde{u}\tilde{u}_{xxxx}))\} = O(\epsilon^3\mu^2, \epsilon^3\mu^4, \mu^6),\end{aligned} \tag{5.7 b}$$

where

$$\alpha \equiv \frac{\tilde{z}}{h} + \frac{1}{2}\left(\frac{\tilde{z}}{h}\right)^2, \tag{5.8}$$

as defined by Nwogu (1993).

(i) *First-order solution*

By substituting (5.6) into (5.7) we get at $O(\epsilon^0)$ two equations of the form (3.13), except for U_1 which is replaced by $\tilde{u}_1$. The coefficients now read

$$\left.\begin{aligned} m_{11}^{(1)} = \omega, \quad m_{12}^{(1)} = -kh(1-(\alpha+\tfrac{1}{3})\kappa^2 + \sigma_1\kappa^4),\\ m_{21}^{(1)} = -k, \quad m_{22}^{(1)} = \omega(1-\alpha\kappa^2 + \sigma_2\kappa^4),\end{aligned}\right\} \tag{5.9}$$

where

$$\sigma_1 \equiv \tfrac{5}{6}(\alpha+\tfrac{2}{5})^2, \quad \sigma_2 \equiv \tfrac{5}{6}\alpha(\alpha+\tfrac{2}{5}). \tag{5.10}$$

Hence, at first order we get the velocity solution,

$$\tilde{u}_1 = \gamma\frac{\omega a_1}{kh}, \quad \gamma \equiv \frac{1}{1-(\alpha+\tfrac{1}{3})\kappa^2 + \sigma_1\kappa^4}, \tag{5.11}$$

and the dispersion relation,

$$\frac{\omega^2}{k^2h} = \frac{1-(\alpha+\tfrac{1}{3})\kappa^2 + \sigma_1\kappa^4}{1-\alpha\kappa^2 + \sigma_2\kappa^4}. \tag{5.12}$$

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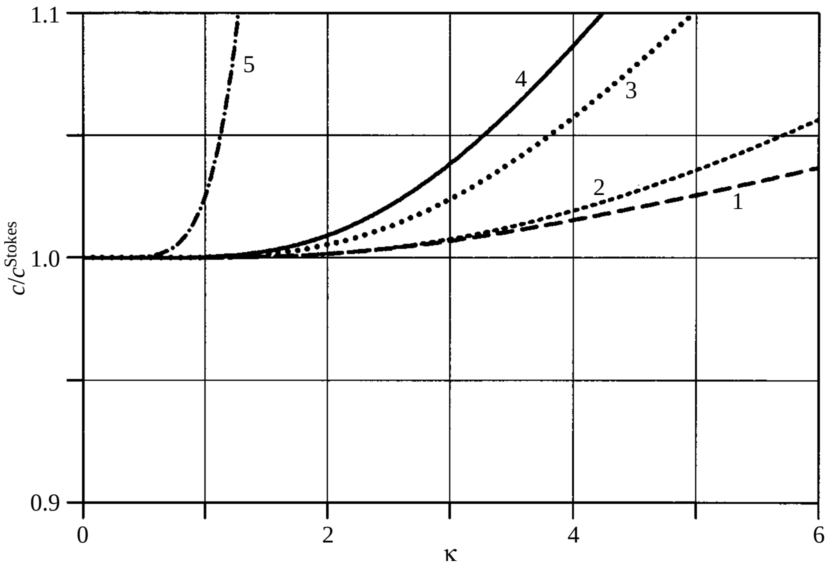

Figure 7. Ratio of phase celerity, c/c^{Stokes}, where c is determined by (5.12) and c^{Stokes} by (3.16). Boussinesq equations include $O(\mu^4)$. (1) $\alpha = -0.429648$; (2) $\alpha = -\frac{4}{9}$; (3) $\alpha = -\frac{2}{5}$; (4) $\alpha = -\frac{1}{2}$; (5) $\alpha = 0$.

The parameter α can be used to optimize the linear dispersion characteristics. As the choice of $\tilde{z}$ defines the velocity variable $\tilde{\boldsymbol{u}}$, we note that according to (5.8) $\tilde{z}$ is inside the still water fluid domain if and only if $-\frac{1}{2} \leqslant \alpha \leqslant 0$.

We note that the choice of $\alpha = -\frac{2}{5}$ provides Padé [2,2] characteristics according to (5.12). For this value of α all the linear μ^4 terms are seen to vanish from (5.7), which demonstrates that the lower-order equations of Nwogu are actually correct to $O(\mu^4)$ on a constant depth provided $\alpha = -\frac{2}{5}$. This result is similar to the one achieved in § 4 *c* with $\alpha_1 = \frac{1}{15}$ and $\beta_1 = 0$, and it confirms again the equivalence between the two alternative enhancement procedures discussed in § 4 *a*.

A comparison of (5.12) and (3.17) indicates the possibility of achieving Padé [4,4] characteristics, but unfortunately the single free parameter α does not allow a perfect matching with (3.17). This would require yet another generalization of the chosen velocity variable, which is not pursued here (see Schäffer & Madsen 1995*b*). One possibility is to choose $\alpha = -\frac{4}{9}$, in which case the κ^2 (but not the κ^4) coefficients can be matched. Another alternative is to minimize the integral,

$$\frac{1}{\kappa_0}\int_0^{\kappa_0}\left(\left(\frac{\omega^2}{k^2 h}\right)^{\text{Stokes}} - \left(\frac{\omega^2}{k^2 h}\right)\right)^2 \mathrm{d}\kappa,$$

which leads to $\alpha = -0.429\,648$ for $\kappa_0 = 6$.

Figure 7 shows the celerity ratio between (5.12) and Stokes's reference solution (3.16) for $\alpha = 0$ (SWL velocity formulation), $\alpha = -\frac{1}{2}$ (bottom velocity formulation), $\alpha = -\frac{2}{5}$ (Padé [2,2]), $\alpha = -\frac{4}{9}$ and $\alpha = -0.429\,648$. Obviously the last choice of α provides the most accurate linear dispersion characteristics, although they are not quite as accurate as the Padé [4,4] characteristics obtained in § 4 *c*.

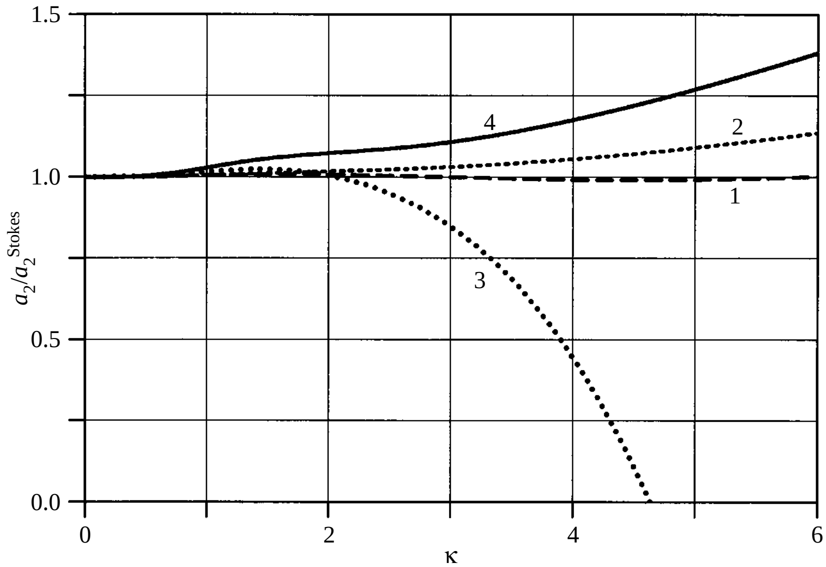

Figure 8. Ratio of second harmonic, a_2/a_2^{Stokes}, where a_2 is determined by (3.20 *a*) combined with (5.13) and a_2^{Stokes} by (3.21). Boussinesq equations include $O(\mu^4, \epsilon\mu^4)$. (1) $\alpha = -0.429\,648$; (2) $\alpha = -\frac{4}{9}$; (3) $\alpha = -\frac{2}{5}$; (4) $\alpha = -\frac{1}{2}$.

(ii) *Second-order solution*

Continuing the analysis to second order and collecting terms of $O(\epsilon)$ leads to the general solution (3.18), except for U_2 which is replaced by $\tilde{u}_2$. The coefficients now read

$$\left.\begin{aligned} m_{11}^{(2)} &= 2\omega, \quad m_{12}^{(2)} = -2kh(1 - 4(\alpha + \tfrac{1}{3})\kappa^2 + 16\sigma_1\kappa^4), \\ m_{21}^{(2)} &= -2k, \quad m_{22}^{(2)} = 2\omega(1 - 4\alpha\kappa^2 + 16\sigma_2\kappa^4), \end{aligned}\right\} \tag{5.13 a}$$

and

$$\left.\begin{aligned} F_1 &= \gamma\omega(1 - \alpha\kappa^2 + \sigma_2\kappa^4), \\ F_2 &= \frac{\gamma^2\omega^2}{2kh}\left(1 - \left(1 + 2\alpha + \frac{2}{\gamma}\right)\kappa^2 + \left(2\sigma_2 + 2(\alpha + \tfrac{1}{3})\left(1 + \frac{1}{\gamma}\right) + \alpha^2\right)\kappa^4\right). \end{aligned}\right\} \tag{5.13 b}$$

Again, the general form of the second-order solution is defined by (3.20), while the target is given by (3.21) and (3.22). By the use of the new coefficients (5.13), an expansion of (3.20 *a*) from $\kappa = 0$ gives

$$a_2 = \frac{3}{4}\frac{a_1^2}{h}\frac{1}{\kappa^2}(1 + c_2\kappa^2 + c_4\kappa^4 + O(\kappa^6)),$$

where $c_2 = \frac{2}{3}$ and $c_4 = \frac{1}{90}(1575\alpha^2 + 1386\alpha + 320)$ when $(\mu^4, \epsilon\mu^4)$ terms are included. In contrast to § 4 *c*, it is not possible to obtain the target value of $c_4 = \frac{7}{45} \approx 0.156$ for any real values of α, because none of them provide Padé [4,4] dispersion characteristics. Instead we get $c_4 = \frac{32}{9} \approx 3.56$ for $\alpha = 0$, $c_4 = \frac{83}{360} \approx 0.231$ for $\alpha = -\frac{1}{2}$, $c_4 = \frac{44}{225} \approx 0.196$ for $\alpha = -\frac{2}{5}$, $c_4 = \frac{68}{405} \approx 0.168$ for $\alpha = -\frac{4}{9}$ and $c_4 = 0.169$ for $\alpha = -0.429648$.

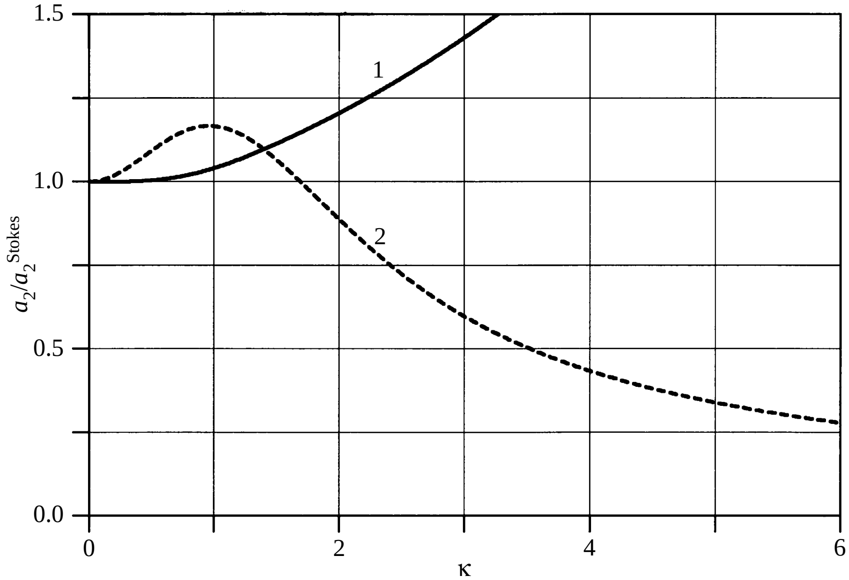

Figure 9. Ratio of second harmonic, a_2/a_2^{Stokes}, where a_2 is determined by (3.20 *a*) combined with (5.13) and a_2^{Stokes} by (3.21). Lower-order Boussinesq equations: (1) Wei *et al*. (1995) include $O(\mu^2, \epsilon\mu^2)$ with $\alpha = -\frac{2}{5}$; (2) Nwogu (1993) include $O(\mu^2, \epsilon)$ with $\alpha = -\frac{2}{5}$.

Table 3. *Expansion coefficients for the second harmonic*

	Stokes	$(\mu^4, \epsilon\mu^4)^{\text{a}}$	$(\mu^2, \epsilon\mu^2)^{\text{b}}$	$(\mu^2, \epsilon)^{\text{b}}$
c_2	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{17}{15}$
c_4	$\frac{7}{45} \approx 0.156$	0.169		

[a] $\alpha = -0.429\,648$, [b] $\alpha = -\frac{2}{5}$.

We note that the case of $\alpha = 0$ is quite poor, and in fact the solution contains a singularity at $\kappa = 1/\sqrt{2}$. Figure 8 shows the ratio between a_2 and the target (3.21) for the four other choices of α. Especially for the choice of $\alpha = -0.429\,648$, the agreement is excellent for $0 \leqslant \kappa \leqslant 6$.

The Boussinesq equations presented by Wei *et al*. (1995) appear as a subset of (5.7) by ignoring the $(\mu^4, \epsilon\mu^4)$ terms. In this case we get $c_2 = -\frac{1}{3}(4 + 15\alpha)$, which leads to the target value of $\frac{2}{3}$ for $\alpha = -\frac{2}{5}$ (corresponding to Padé [2,2] dispersion characteristics). A further neglect of $\epsilon\mu^2$ terms corresponds to the equations of Nwogu (1993) and results in $c_2 = -\frac{1}{3}(1 + 11\alpha)$, i.e. $\frac{17}{15}$ for $\alpha = -\frac{2}{5}$. This confirms the conclusion from § 4 *c* that the potential of an improvement of the linear dispersion characteristics (in this case Padé [2,2] achieved with $\alpha = -\frac{2}{5}$) is used fully with regard to the nonlinear transfer only if a sufficient order of dispersion is retained also in the nonlinear terms (in this case $\epsilon\mu^2$ terms versus ϵ terms). As seen from figure 9, the equations of Wei *et al*. are clearly superior to the equations of Nwogu for $\kappa \leqslant 1.5$. On the other hand, the equations of Wei *et al*. are not particularly attractive for larger values of κ.

(iii) *Third-order solution*

We continue the Fourier analysis of (5.7) to third order and collect terms of order $O(\epsilon^2)$. Secular $\sin\theta$ terms are removed by introducing

$$\omega = \omega_1(1+\epsilon^2\omega_{13}), \quad \tilde{u}_1 = \gamma(\omega a_1/kh)(1+\epsilon^2\tilde{u}_{13}), \tag{5.14}$$

where γ is defined by (5.11), ω_{13} represents the amplitude dispersion and $\tilde{u}_{13}$ the third-order correction to the first-order velocity. Again we focus on the quantities a_3 and ω_{13} for which the Stokes reference solutions are given by (3.25). The ratios of a_3 and ω_{13} to the target solutions are shown as a function of κ in figure 10 for four different values of α. Again we have omitted the case of $\alpha = 0$ as it contains singularities at $\kappa = \frac{1}{2}$ and $\kappa = 1/\sqrt{2}$. The solutions obtained for the case of $\alpha = -0.429\,648$ are again quite satisfactory.

As discussed previously, the lower-order equations of Wei *et al*. (1995) and Nwogu (1993) appear as a subset of (5.7) and the third-order analysis performed above can easily be adjusted to their equations. The resulting variation of the ratios of a_3 and ω_{13} to the target solutions are shown as a function of κ in figure 11 for $\alpha = -\frac{2}{5}$. Again the $(\mu^2, \epsilon\mu^2, \epsilon^2\mu^2)$ equations of Wei *et al*. (1995) are seen to perform better than the (μ^2, ϵ) equations of Nwogu (1993), especially for small values of κ. In both cases, however, the result is less attractive than the results obtained in figure 10 with $\alpha = -0.429\,648$.

6. Enhanced $(\mu^2, \epsilon^3\mu^2)$ equations in terms of the velocity at an arbitrary z location, $\tilde{\boldsymbol{u}}$

(*a*) *Derivation of enhanced equations*

When the velocity is taken at an arbitrary z location, the equations can be enhanced to get excellent dispersion even when the μ^4 terms are neglected. Schäffer & Madsen (1995*a*) presented a procedure by which the lower-order (μ^2, ϵ) equations of Nwogu (1993) were enhanced to incorporate linear dispersion characteristics corresponding to a Padé [4,4] expansion (see Schröter *et al*. (1994) for a similar derivation). In this section we shall apply the same procedure on the $(\mu^2, \epsilon^3\mu^2)$ equations of Wei *et al*. (1995), and demonstrate that the enhanced equations incorporate improved nonlinearity as well as Padé [4,4] dispersion characteristics. In § 8 we shall show that these characteristics can also be obtained in the case of Doppler shift in connection with wave–current interaction.

As in § 4 *b* we introduce four free parameters $(\alpha_1, \alpha_2, \beta_1, \beta_2)$ which are of order $O(1)$. First, we modify the momentum equation by using the operators

$$(\alpha_2-\alpha_1)\mu^2 h^2\nabla(\nabla\cdot \quad) \quad \text{and} \quad -\alpha_2\mu^2 h\nabla(\nabla\cdot h \quad),$$

respectively, on (5.4). This yields

$$(\alpha_2-\alpha_1)\mu^2 h^2\{\nabla(\nabla\cdot\tilde{\boldsymbol{u}}_t) + \nabla(\nabla^2\eta) + \tfrac{1}{2}\epsilon\nabla(\nabla\cdot(\nabla(\tilde{\boldsymbol{u}})^2))\} = O(\mu^4, \epsilon\mu^4), \tag{6.1 a}$$

$$-\alpha_2\mu^2 h\{\nabla(\nabla\cdot(h\tilde{\boldsymbol{u}}_t)) + \nabla(\nabla\cdot(h\nabla\eta)) + \tfrac{1}{2}\epsilon\nabla(\nabla\cdot(h\nabla(\tilde{\boldsymbol{u}})^2))\} = O(\mu^4, \epsilon\mu^4). \tag{6.1 b}$$

Second, a similar procedure is followed in connection with the continuity equation. In this case we use the operators $(\beta_2-\beta_1)\mu^2\nabla\cdot(h^2\nabla \quad)$ and $-\beta_2\mu^2\nabla^2(h^2 \quad)$, respec-

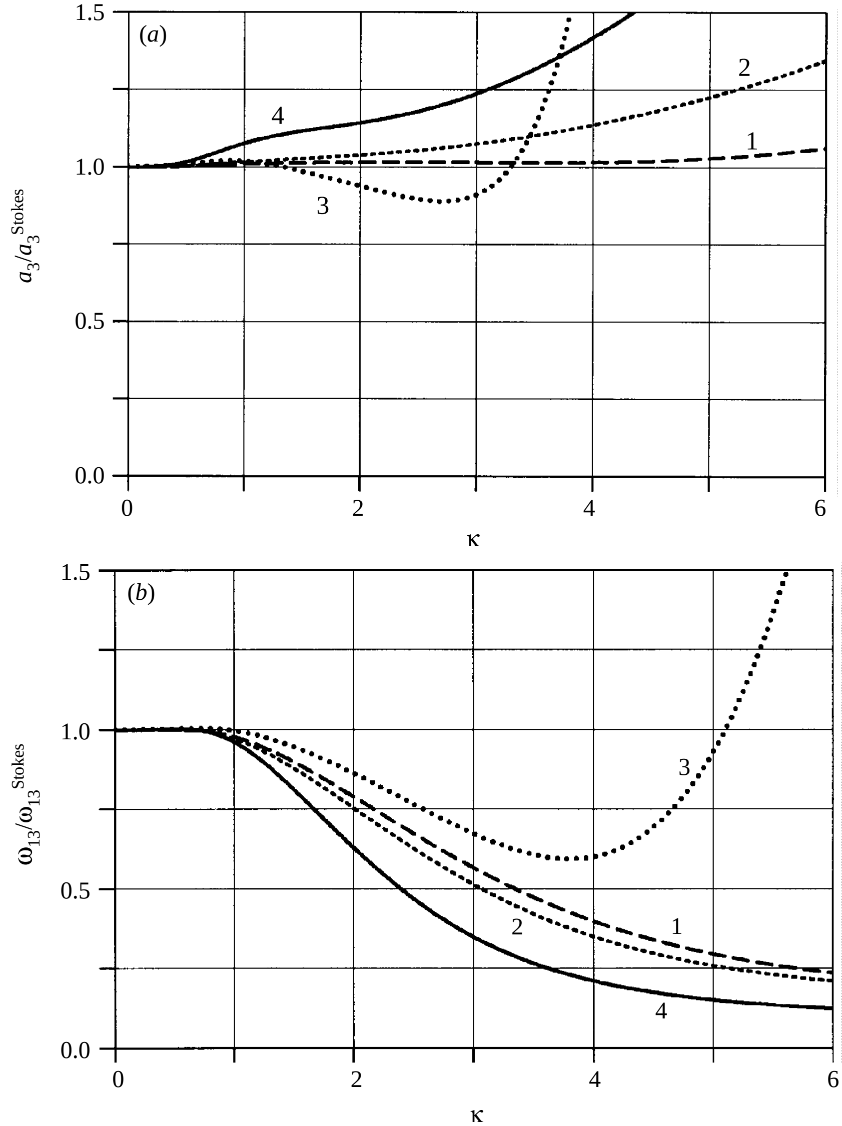

Figure 10. (*a*) Ratio of third harmonic, a_3/a_3^{Stokes}, where a_3^{Stokes} is given by (3.25). Boussinesq equations include $O(\mu^4, \epsilon\mu^4, \epsilon^2\mu^4)$. (1) $\alpha = -0.429\,648$; (2) $\alpha = -\frac{4}{9}$; (3) $\alpha = -\frac{2}{5}$; (4) $\alpha = -\frac{1}{2}$. (*b*) Ratio of amplitude dispersion, $\omega_{13}/\omega_{13}^{\text{Stokes}}$, where $\omega_{13}^{\text{Stokes}}$ is given by (3.25). Boussinesq equations include $O(\mu^4, \epsilon\mu^4, \epsilon^2\mu^4)$. (1) $\alpha = -0.429\,648$; (2) $\alpha = -\frac{4}{9}$; (3) $\alpha = -\frac{2}{5}$; (4) $\alpha = -\frac{1}{2}$.

tively, on (2.18) with $\boldsymbol{Q}$ defined by (5.3) and obtain

$$(\beta_2 - \beta_1)\mu^2\nabla \cdot [h^2\nabla\eta_t + h^2\nabla(\nabla \cdot ((h+\epsilon\eta)\tilde{\boldsymbol{u}}))] = O(\mu^4, \epsilon\mu^4), \tag{6.2 a}$$

$$-\beta_2\mu^2[\nabla^2(h^2\eta_t) + \nabla^2(h^2\nabla \cdot ((h+\epsilon\eta)\tilde{\boldsymbol{u}}))] = O(\mu^4, \epsilon\mu^4). \tag{6.2 b}$$

By adding (6.2) to (2.18) with (5.3) and (6.1) to (5.4), we obtain the following enhanced continuity and momentum equations retaining all nonlinear terms at order

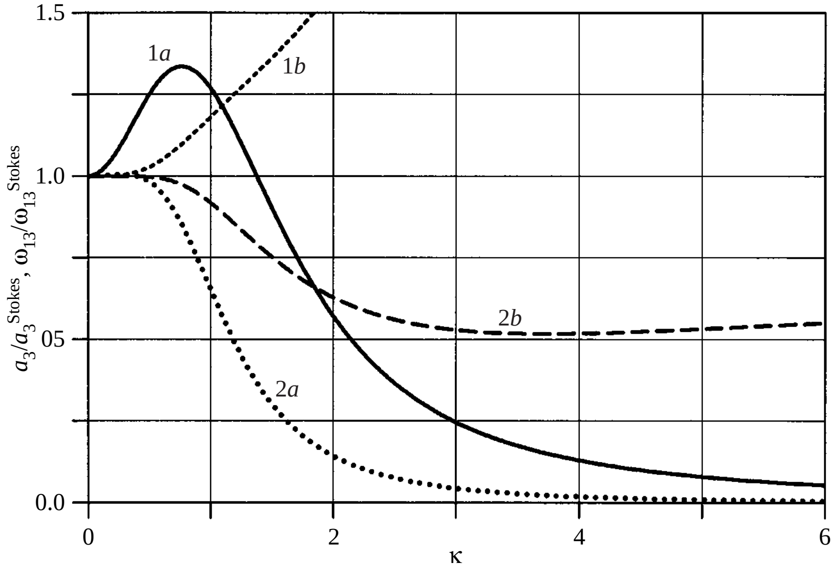

Figure 11. Ratios of (1a), (1b) a_3/a_3^{Stokes} and (2a), (2b) $\omega_{13}/\omega_{13}^{\text{Stokes}}$. (1$a$), (2$a$) Nwogu (1993) equations include $O(\mu^2, \epsilon)$ with $\alpha = -\frac{2}{5}$. (1b), (2b) Wei *et al.* (1995) equations include $O(\mu^2, \epsilon\mu^2, \epsilon^2\mu^2)$ with $\alpha = -\frac{2}{5}$.

$O(\mu^2)$:

$$\eta_t + \nabla \cdot \boldsymbol{Q} = O(\mu^4), \tag{6.3 a}$$

$$\tilde{\boldsymbol{u}}_t + \nabla\eta + \tfrac{1}{2}\epsilon\nabla(\tilde{\boldsymbol{u}})^2 + \mu^2[\Lambda_{20}^{\text{V}} + \epsilon\Lambda_{21}^{\text{V}} + \epsilon^2\Lambda_{22}^{\text{V}} + \epsilon^3\Lambda_{23}^{\text{V}}] = O(\mu^4), \tag{6.3 b}$$

where

$$\begin{aligned}\boldsymbol{Q} &= \tilde{\boldsymbol{u}}(h+\epsilon\eta) + \mu^2[(\tfrac{1}{2}h^2 + h\tilde{z} + \epsilon\eta\tilde{z} - \tfrac{1}{2}\epsilon^2\eta^2)\nabla(\nabla\cdot(h\tilde{\boldsymbol{u}}))\\ &\quad + (\tfrac{1}{2}h\tilde{z}^2 - \tfrac{1}{6}h^3 + \tfrac{1}{2}\epsilon\eta\tilde{z}^2 - \tfrac{1}{6}\epsilon^3\eta^3)\nabla(\nabla\cdot\tilde{\boldsymbol{u}}) + (\beta_2-\beta_1)h^2\nabla(\nabla\cdot((h+\epsilon\eta)\tilde{\boldsymbol{u}}))\\ &\quad - \beta_2\nabla(h^2\nabla\cdot((h+\epsilon\eta)\tilde{\boldsymbol{u}})) + (\beta_2-\beta_1)h^2\nabla\eta_t - \beta_2\nabla(h^2\eta_t)],\end{aligned} \tag{6.4 a}$$

$$\begin{aligned}\Lambda_{20}^{\text{V}} &= (\tilde{z} - \alpha_2 h)\nabla(\nabla\cdot(h\tilde{\boldsymbol{u}}_t)) + (\tfrac{1}{2}\tilde{z}^2 + (\alpha_2-\alpha_1)h^2)\nabla(\nabla\cdot\tilde{\boldsymbol{u}}_t)\\ &\quad + (\alpha_2-\alpha_1)h^2\nabla(\nabla^2\eta) - \alpha_2 h\nabla(\nabla\cdot(h\nabla\eta)),\end{aligned} \tag{6.4 b}$$

$$\begin{aligned}\Lambda_{21}^{\text{V}} &= \nabla(\tilde{\boldsymbol{u}}\cdot(\tilde{z}\nabla(\nabla\cdot(h\tilde{\boldsymbol{u}})) + \tfrac{1}{2}\tilde{z}^2\nabla(\nabla\cdot\tilde{\boldsymbol{u}})) - \eta\nabla\cdot(h\tilde{\boldsymbol{u}}_t) + \tfrac{1}{2}(\nabla\cdot(h\tilde{\boldsymbol{u}}))^2)\\ &\quad + \tfrac{1}{2}(\alpha_2-\alpha_1)h^2\nabla(\nabla^2(\tilde{\boldsymbol{u}}^2)) - \tfrac{1}{2}\alpha_2 h\nabla(\nabla\cdot(h\nabla(\tilde{\boldsymbol{u}}^2))),\end{aligned} \tag{6.4 c}$$

and where $\Lambda_{22}^{\text{V}} = \Lambda_{22}^{\text{IV}}$ and $\Lambda_{23}^{\text{V}} = \Lambda_{23}^{\text{IV}}$ are defined by (5.5 c), (5.5 d). The four coefficients α_1, α_2, β_1 and β_2 are yet to be determined. As in § 4, the set (α_1, β_1) governs the linear dispersion relation, while the set (α_2, β_2) can be used to optimize the linear shoaling gradient.

(b) *Fourier analysis of equations on a horizontal bottom*

The analysis of the dispersion and nonlinearity characteristics will follow the procedure outlined in §§ 3 b, 4 c and 5 b. In one dimension and on a horizontal bottom, (6.3) simplify to

$$\eta_t + h\tilde{u}_x + \mu^2((\alpha + \tfrac{1}{3} - \beta_1)h^3\tilde{u}_{xxx} - \beta_1 h^2 \eta_{xxt}) + \epsilon(\eta\tilde{u})_x + \epsilon\mu^2 \frac{\partial}{\partial x}(\alpha h^2 \eta \tilde{u}_{xx} - \beta_1 h^2(\eta\tilde{u})_{xx}) - \epsilon^2\mu^2 \frac{\partial}{\partial x}(\tfrac{1}{2}h\eta^2\tilde{u}_{xx}) = O(\epsilon^3\mu^2, \mu^4), \quad (6.5\,a)$$

$$\tilde{u}_t + \eta_x + \mu^2((\alpha - \alpha_1)h^2\tilde{u}_{xxt} - \alpha_1 h^2 \eta_{xxx}) + \epsilon\tilde{u}\tilde{u}_x + \epsilon\mu^2\frac{\partial}{\partial x}(\alpha h^2 \tilde{u}\tilde{u}_{xx} - \tfrac{1}{2}\alpha_1 h^2(\tilde{u}^2)_{xx} + \tfrac{1}{2}h^2(\tilde{u}_x)^2 - h\eta\tilde{u}_{xt}) + \epsilon^2\mu^2\frac{\partial}{\partial x}(-\tfrac{1}{2}\eta^2\tilde{u}_{xt} + h\eta(\tilde{u}_x^2 - \tilde{u}\tilde{u}_{xx})) = O(\epsilon^3\mu^2, \mu^4), \quad (6.5\,b)$$

where α is defined by (5.8). We look for third-order solutions to (6.5) of the form (5.6).

(i) *First-order solution*

Collecting terms of $O(\epsilon^0)$ leads to (3.13) with coefficients determined by

$$\left.\begin{aligned} m_{11}^{(1)} &= \omega(1 + \beta_1\kappa^2), \quad m_{12}^{(1)} = -kh(1 - (\alpha + \tfrac{1}{3} - \beta_1)\kappa^2), \\ m_{21}^{(1)} &= -k(1 + \alpha_1\kappa^2), \quad m_{22}^{(1)} = \omega(1 - (\alpha - \alpha_1)\kappa^2). \end{aligned}\right\} \quad (6.6)$$

Hence, at first order we get the velocity solution

$$\tilde{u}_1 = \gamma\frac{\omega a_1}{kh}, \quad \gamma \equiv \frac{1 + \beta_1\kappa^2}{1 - (\alpha + \frac{1}{3} - \beta_1)\kappa^2} \quad (6.7)$$

and the dispersion relation

$$\frac{\omega^2}{k^2 h} = \frac{1 + (\alpha_1 + \beta_1 - \alpha - \frac{1}{3})\kappa^2 + \alpha_1(\beta_1 - \alpha - \frac{1}{3})\kappa^4}{1 + (\alpha_1 + \beta_1 - \alpha)\kappa^2 + \beta_1(\alpha_1 - \alpha)\kappa^4}. \quad (6.8)$$

By matching (6.8) with the Padé [4,4] expansion given by (3.17), Schäffer & Madsen (1995*a*) determined the following four sets of solutions for $(\alpha, \beta_1, \alpha_1)$:

$$(\alpha, \beta_1, \alpha_1) = (\tfrac{1}{18}(-3 - \sqrt{\tfrac{23}{35}} - 2\sqrt{\tfrac{19}{7}}), \tfrac{1}{126}(28 - 2\sqrt{133}), \tfrac{1}{1890}(105 - 3\sqrt{805})) \approx (-0.395, 0.039, 0.011), \quad (6.9\,a)$$

$$(\alpha, \beta_1, \alpha_1) = (\tfrac{1}{18}(-3 + \sqrt{\tfrac{23}{35}} - 2\sqrt{\tfrac{19}{7}}), \tfrac{1}{126}(28 - 2\sqrt{133}), \tfrac{1}{1890}(105 + 3\sqrt{805})) \approx (-0.305, 0.039, 0.101), \quad (6.9\,b)$$

$$(\alpha, \beta_1, \alpha_1) = (\tfrac{1}{18}(-3 - \sqrt{\tfrac{23}{35}} + 2\sqrt{\tfrac{19}{7}}), \tfrac{1}{126}(28 + 2\sqrt{133}), \tfrac{1}{1890}(105 - 3\sqrt{805})) \approx (-0.029, 0.405, 0.011), \quad (6.9\,c)$$

$$(\alpha, \beta_1, \alpha_1) = (\tfrac{1}{18}(-3 + \sqrt{\tfrac{23}{35}} + 2\sqrt{\tfrac{19}{7}}), \tfrac{1}{126}(28 + 2\sqrt{133}), \tfrac{1}{1890}(105 + 3\sqrt{805})) \approx (0.061, 0.405, 0.101). \quad (6.9\,d)$$

Notice that in set I the value of α is only slightly different from the $-\frac{2}{5}$ considered in § 5 *b*, while set IV contains a small positive value of α, indicating a velocity variable taken above the SWL. All four sets provide the excellent linear dispersion characteristics represented by figure 3.

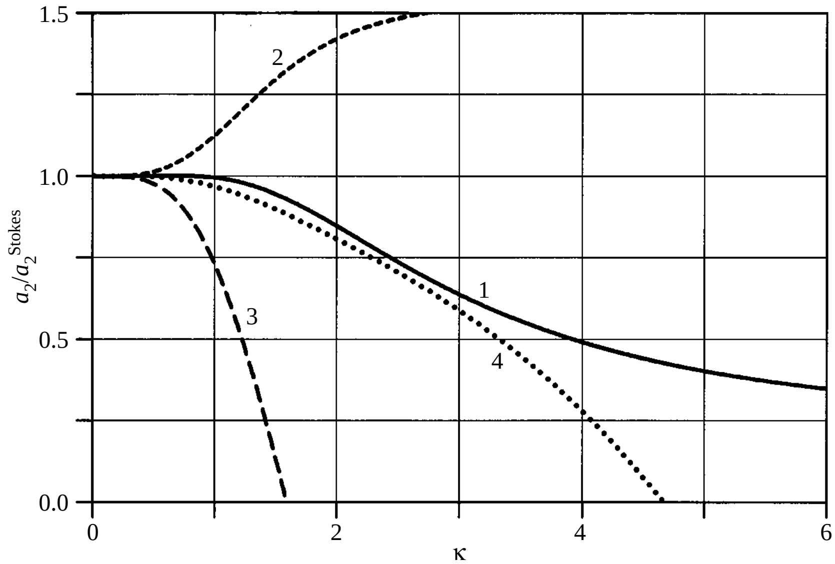

Figure 12. Ratio of second harmonic, a_2/a_2^{Stokes}, where a_2 is determined by (3.20 *a*) combined with (6.10) and a_2^{Stokes} by (3.21). Enhanced equations include $O(\mu^2, \epsilon\mu^2)$; (1) set I, i.e. (6.9 *a*); (2) set II, i.e. (6.9 *b*); (3) set III, i.e. (6.9 *c*); (4) set IV, i.e. (6.9 *d*).

(ii) *Second-order solution*

Continuing the analysis to second order and collecting terms of $O(\epsilon)$ leads to the general algebraic problem of (3.18) with coefficients determined by

$$\left.\begin{aligned} m_{11}^{(2)} &= 2\omega(1+4\beta_1\kappa^2), \quad m_{12}^{(2)} = -2kh(1-4(\alpha+\tfrac{1}{3}-\beta_1)\kappa^2), \\ m_{21}^{(2)} &= -2k(1+4\alpha_1\kappa^2), \quad m_{22}^{(2)} = 2\omega(1-4(\alpha-\alpha_1)\kappa^2), \end{aligned}\right\} \tag{6.10 a}$$

$$F_1 = \gamma\omega(1+(4\beta_1-\alpha)\kappa^2), \quad F_2 = \frac{\gamma^2\omega^2}{2kh}\left(1+\left(4\alpha_1-1-2\alpha-\frac{2}{\gamma}\right)\kappa^2\right). \tag{6.10 b}$$

Again the general form of the second-order solution is defined by (3.20). An expansion of (3.20 *a*) from $\kappa = 0$ with coefficients defined by (6.10) leads to

$$a_2 = \frac{3}{4}\frac{a_1^2}{h}\frac{1}{\kappa^2}(1+c_2\kappa^2+c_4\kappa^4+O(\kappa^6)),$$

where $c_2 = \frac{1}{3}(45\lambda - 4)$ with $\lambda \equiv \alpha(\beta_1-\alpha_1)+\frac{1}{3}(\beta_1-\alpha)$. It turns out that a polynomial expansion from $\kappa = 0$ (Padé [4,0] type) of the dispersion relation (6.8) yields a κ^4 factor which is identical to λ as defined above. The target value of this factor is $\frac{2}{15}$ (according to Stokes's linear dispersion relation) and this value is also obtained for all four sets of $(\alpha, \beta_1, \alpha_1)$ given by (6.9). With this value of λ the above expression for c_2 becomes identical to the target value of $\frac{2}{3}$. Hence again it can be concluded that the enhancement of the linear dispersion characteristics has a positive effect also on the nonlinear properties of the equations. As discussed in § 5 *b*, the case of $(\alpha, \beta_1, \alpha_1) = (-\frac{2}{5}, 0, 0)$ also leads to the target value of c_2. None of the sets leads to the target value of $\frac{7}{45} \approx 0.156$ for c_4 and we get the numerical values $c_4 = 0.1808$ (I),

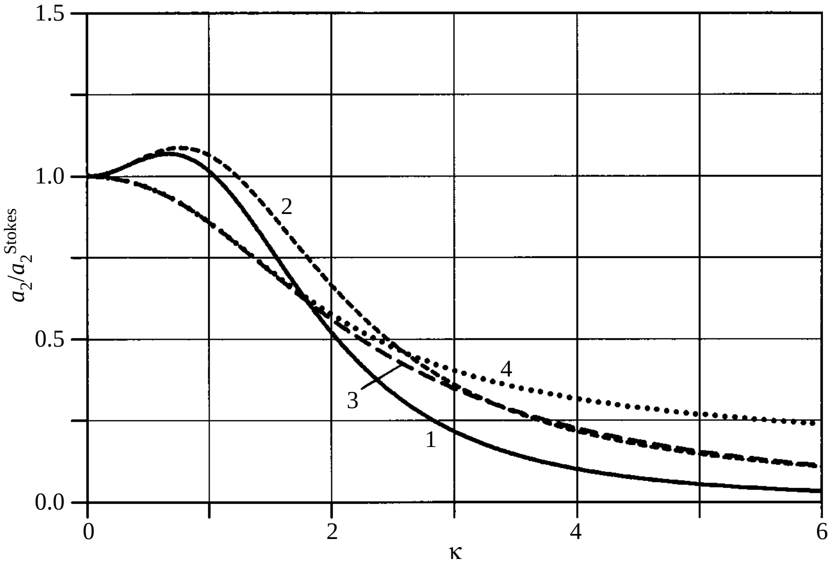

Figure 13. Ratio of second harmonic, a_2/a_2^{Stokes}, where a_2 is determined by (3.20 *a*) combined with (6.10) and a_2^{Stokes} by (3.21). Enhanced equations include $O(\mu^2, \epsilon)$; (1) set I, i.e. (6.9 *a*); (2) set II, i.e. (6.9 *b*); (3) set III, i.e. (6.9 *c*); (4) set IV, i.e. (6.9 *d*).

Table 4. *Expansion coefficients for the second harmonic including terms of* $O(\mu^2, \epsilon\mu^2)$

	Stokes	I (6.9 *a*)	II (6.9 *b*)	III (6.9 *c*)	IV (6.9 *d*)
c_2	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$
c_4	$\frac{7}{45}$	0.1808	0.4161	−0.2402	0.1049

0.4161 (II), −0.2402 (III) and 0.1049 (IV). This indicates that set I given by (6.9 *a*) is the most attractive one, and this is confirmed by figure 12 showing the variation of a_2 relative to (3.21) as a function of κ: only moderate discrepancies occur for κ less than *ca.* 2.

The enhanced equations of Schäffer & Madsen (1995*a*) appear as a subset of (6.5) by neglecting the $\epsilon\mu^2$ terms. Without these terms it is no longer possible to obtain the correct coefficient to c_2 and the variation of a_2 is clearly less accurate than before. Figure 13 shows this variation for the four sets of $(\alpha, \beta_1, \alpha_1)$ defined by (6.9) and we notice that the accuracy is now similar to what was achieved by the equations of Nwogu (see figure 9). This, once again, confirms the conclusion that the potential of the linear improvement is fully utilized with regard to the second-order transfer only if sufficient order of dispersion is retained also in the nonlinear terms.

(iii) *Third-order solution*

We proceed as described in § 5, where γ in (5.14) is now defined by (6.7). Again we focus on the quantities a_3 and ω_{13} for which the target solution is given in (3.25). It turns out that by far the best performance of these quantities is obtained for the coefficient set I in (6.9 *a*), which confirms the conclusions based on the second-order

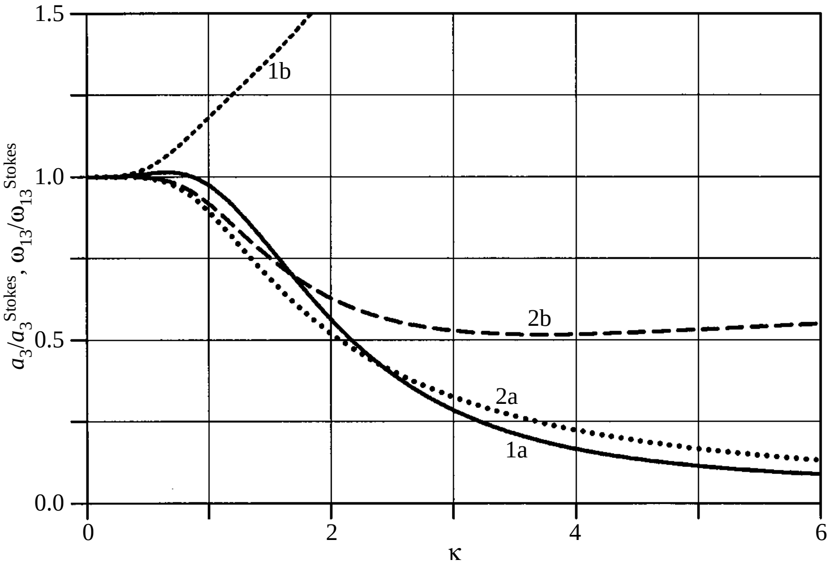

Figure 14. Ratios of (1a) ,(1b) a_3/a_3^{Stokes} and (2a), (2b) $\omega_{13}/\omega_{13}^{\text{Stokes}}$. (1$a$), (2$a$) Enhanced equations include $O(\mu^2, \epsilon\mu^2, \epsilon^2\mu^2)$ with parameter set I (6.9 a); (1b), (2b) Wei *et al*. (1995) equations include $O(\mu^2, \epsilon\mu^2, \epsilon^2\mu^2)$ with $\alpha = -\frac{2}{5}$.

solution. The ratios of a_3 and ω_{13} to the target solutions are shown as a function of κ in figure 14 for the parameter set I defined in (6.9 a). As a reference, figure 14 includes the solutions corresponding to Wei *et al*. (1995) (also shown in figure 11). We notice that the new enhanced equations perform better with respect to a_3 and somewhat worse with respect to ω_{13}. However, in practice, the amplitude dispersion will be significantly influenced by the frequency dispersion, and for this reason the enhanced equations (with Padé [4,4] characteristics) are to be preferred in comparison to Wei *et al*. (1995) (with Padé [2,2] characteristics).

(c) *Linear shoaling analysis*

The linear shoaling analysis follows the procedure outlined in § 4 d and again the remaining two free parameters (β_2, α_2) are determined from minimization of the integral,

$$\frac{1}{\kappa_0}\int_0^{\kappa_0} (\gamma_0^{\text{Stokes}} - \gamma_0)^2 \, \mathrm{d}\kappa.$$

This analysis was presented by Schäffer & Madsen (1995a), who found that the best linear shoaling properties could be achieved on the basis of set III of $(\alpha, \beta_1, \alpha_1)$. As discussed in § 6 b we do not recommend this set in the present work, because the nonlinear properties are much better in the case of set I (6.9 a), while this set still allows for excellent linear shoaling properties. On the basis of $(\alpha, \beta_1, \alpha_1)$ given by (6.9 a) and with $\kappa_0 = 6$, we determine a minimum of 5.7×10^{-3} of the above integral for $(\beta_2, \alpha_2) = (0.144\,53, 0.021\,53)$.

7. Transfer functions for sub- and superharmonics

(*a*) *Introduction*

In this section transfer functions for second-order bound sub- and superharmonics are derived on the basis of the new enhanced Boussinesq equations presented in §§ 4 and 6, respectively. The accuracy is tested against the exact expressions derived from the nonlinear boundary value problem for the Laplace equation. Furthermore, comparison is made with results for several other sets of equations known from the literature. Most of these appear as special cases of the equations from §§ 4 and 6. Throughout the comparisons it should be kept in mind that the accuracy of second-order transfer is by far most important in shallow water where the energy transfer is large.

We consider on a constant depth the forcing due to a simple first-order wave group made up of just two frequencies ω_n and ω_m,

$$\eta(x,t) = \eta_n + \eta_m, \tag{7.1 a}$$

$$\eta_n = a_n \cos(\omega_n t - k_n x) + b_n \sin(\omega_n t - k_n x), \tag{7.1 b}$$

$$\eta_m = a_m \cos(\omega_m t - k_m x) + b_m \sin(\omega_m t - k_m x), \tag{7.1 c}$$

Each of the two wave components are considered to be solutions to the relevant linearized Boussinesq equations, and consequently both sets (ω_n, k_n) and (ω_m, k_m) satisfy the corresponding linear dispersion relation.

The nonlinear terms of order $O(\epsilon)$ are quadratic, and through these terms a first-order bichromatic wave train will force a second-order wave train consisting of four contributions, one subharmonic, $\omega_p = \omega_n - \omega_m$, and three superharmonics, $\omega_p = \omega_n + \omega_m$, $\omega_p = 2\omega_n$ and $\omega_p = 2\omega_m$ with corresponding wavenumbers determined by $k_p = k_n - k_m$, $k_p = k_n + k_m$, $k_p = 2k_n$ and $k_p = 2k_m$. These waves are bound or phase-locked to the first-order wave train and (ω_p, k_p) does not satisfy the linear dispersion relation. We can express the second-order wave train by

$$\eta^{(2)}(x,t) = \eta^-_{nm} + \eta^+_{nm} + \eta^+_{nn} + \eta^+_{mm}, \tag{7.2}$$

where

$$\eta^\pm_{nm} = \epsilon\delta G^\pm_\eta (a_p \cos(\omega_p t - k_p x) + b_p \sin(\omega_p t - k_p x)), \tag{7.3 a}$$

and where

$$a_p = \frac{1}{h}(a_n a_m \mp b_n b_m), \quad b_p = \frac{1}{h}(a_m b_n \pm a_n b_m), \tag{7.3 b}$$

$$\omega_p = \omega_n \pm \omega_m, \quad k_p = k_n \pm k_m \tag{7.3 c}$$

and

$$\delta = \begin{cases} \frac{1}{2} & \text{for } n = m, \\ 1 & \text{for } n \neq m. \end{cases} \tag{7.3 d}$$

Notice that the sub/superharmonic contributions in (7.2) are found by using the lower/upper signs in (7.3 *a*)–(7.3 *c*). The G^-_η and G^+_η are the second-order surface elevation transfer functions, which are to be determined on the basis of the particular Boussinesq-type equations in the two following sections. Expressions for the second-order velocity can be obtained by using $G^\pm_u$ instead of $G^\pm_\eta$ in (7.3 *a*).

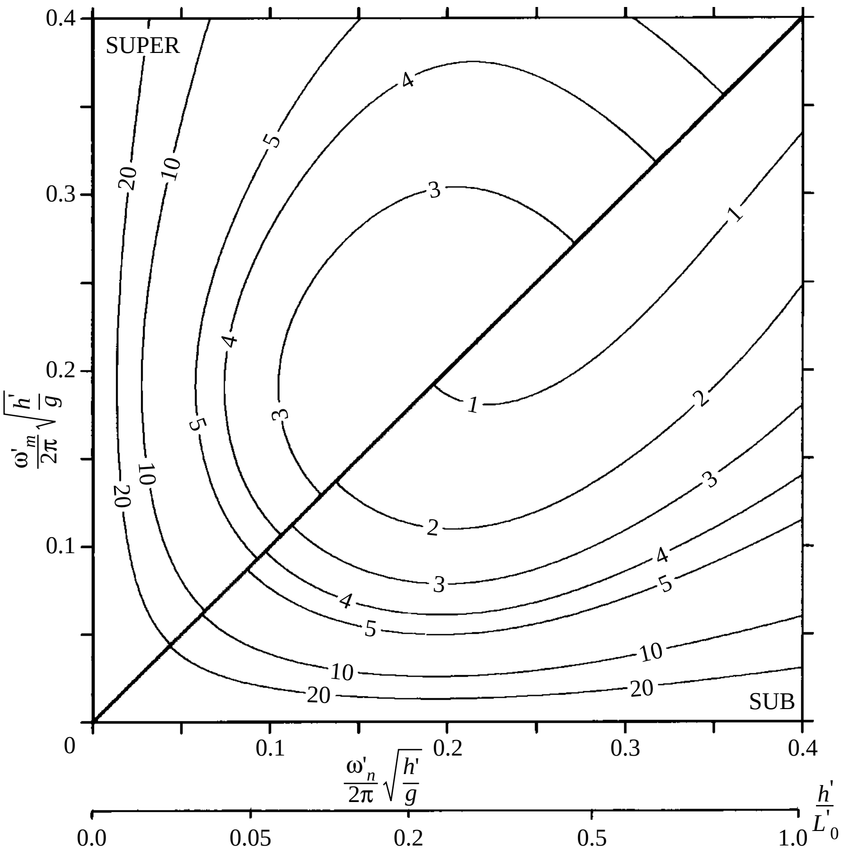

Figure 15. Second-order transfer function G_η^{Stokes} for Stokes theory. Superharmonic result, G_η^+, shown above the diagonal; subharmonic result, $-G_\eta^-$, shown below it.

The target transfer function $G_\eta^\pm$, determined directly from the nonlinear boundary value problem for the Laplace equation, was given by, for example, Ottesen-Hansen (1978) for the subharmonics and Sand & Mansard (1986) for the superharmonics (see also Dean & Sharma 1981). Figure 15 shows the variation of $G_\eta^\pm$ as a function of Ω_n and Ω_m defined by

$$\Omega_n \equiv \frac{\omega_n'}{2\pi}\sqrt{\frac{h'}{g}}, \quad \Omega_m \equiv \frac{\omega_m'}{2\pi}\sqrt{\frac{h'}{g}}.$$

A corresponding axis is shown in terms of the parameter h'/L_0', where L_0 is the linear deep water wavelength. The upper triangle in figure 15 represents the superharmonic transfer from ω_n and ω_m to $\omega_p = \omega_n + \omega_m$, while the lower triangle represents the subharmonic transfer from ω_n and ω_m to $\omega_p = \omega_n - \omega_m$. The diagonal line on which ω_n equals ω_m represents the second-harmonic transfer, which was discussed in §§ 3 *b*, 4 *c*, 5 *b* and 6 *b*.

From figure 15 we notice that the superharmonic transfer has a local minimum in intermediate water depth, while the subharmonic is gradually reduced going from shallow to deep water. Both functions go to infinity in the shallow water limit, where the difference between bound and free wavenumbers vanish. Approaching this limit

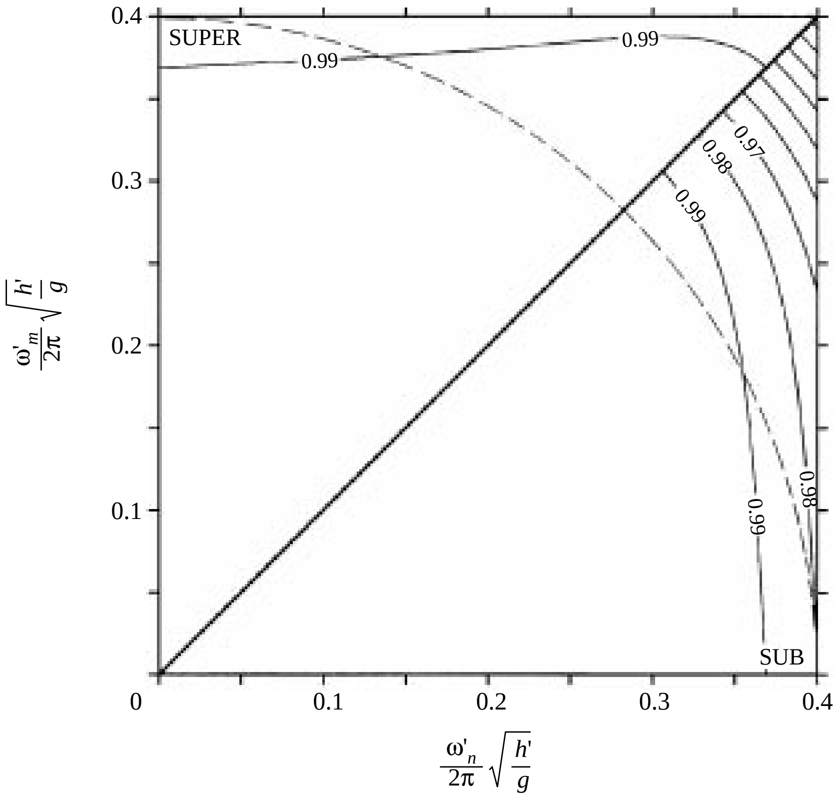

Figure 16. Ratio of wavenumber for bound wave, $(k_n \pm k_m)/(k_n^{\text{Stokes}} \pm k_m^{\text{Stokes}})$. Results for dispersion relation corresponding to (*a*) Padé [4,4]. The dashed curve indicates a 2% error circle as defined in the text.

the generated bound waves will no longer be small compared to the primary waves and near-resonant energy exchange will take place. For a further discussion of this aspect see, for example, Madsen & Sørensen (1993).

When evaluating the second-order transfer for a given set of equations, not only the amplitude is relevant. For the subharmonics in particular, also the speed of propagation or the wavenumber of the bound component is important (e.g. for resonance in harbours). The quality of the equations from §§ 4 and 6, respectively, is the same in this respect, since both sets correspond to a Padé [4,4] dispersion relation governing the sum and difference wavenumbers for the second-order components. Using the convention of figure 15 with respect to axes and sub/superharmonics, figure 16 shows the sum and difference wavenumbers relative to the results from Stokes's dispersion relation, i.e. $(k_n + k_m)/(k_n^{\text{Stokes}} + k_m^{\text{Stokes}})$ above and $(k_n - k_m)/(k_n^{\text{Stokes}} - k_m^{\text{Stokes}})$ below the diagonal. For reference, the equivalent result is shown in figure 16*b* for a Padé [2,2] dispersion relation. Approaching the diagonal from above, the result equals the reciprocal of the relative celerity error, i.e. $(c/c^{\text{Stokes}})^{-1}$ for $\omega = \omega_n = \omega_m$. Approaching the diagonal from below the same comment applies, but for the group velocity.

Both contour plots in figure 16 include a 2% 'error circle' centred at $(\Omega_n, \Omega_m) = (0, 0)$ and defined by the maximum radius $\Omega_{\text{R}} \equiv \sqrt{(\Omega_n^2 + \Omega_m^2)}$ for which the error is

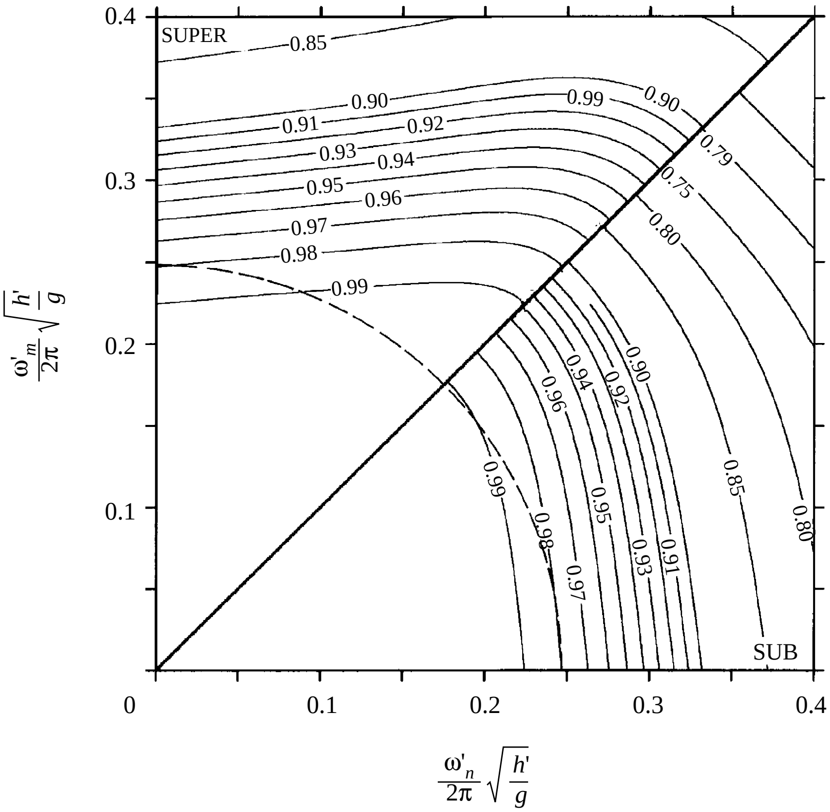

Figure 16. (*Cont.*) Results for dispersion relation corresponding to (*b*) Padé [2,2].

less than 2% for any set of (Ω_n, Ω_m). This simplified way of illustrating the error is convenient in the comparisons of the transfer functions given below. The justification for using a circle instead of, for example, a square with $\Omega_{\mathrm{R}} \equiv \max(\Omega_n, \Omega_m)$ is that the tail of primary-wave spectra will normally result in very weak interaction when both primary frequencies are high.

(*b*) *Analysis of* $\boldsymbol{U}$ *formulation from § 4*

In this section we shall derive the $G_\eta^\pm$ functions on the basis of the new enhanced higher-order Boussinesq equations formulated in the depth-averaged velocity $\boldsymbol{U}$ in (4.2) and (3.2). In one dimension and on a horizontal bottom, (4.2) and (3.2) simplify to (4.4), which will be used as a starting point for the derivation.

As (7.1) is a solution to the linearized equations, (ω_n, k_n) and (ω_m, k_m) will satisfy the dispersion relation (4.6). Furthermore, the first-order velocity can be expressed by

$$U^{(1)}(x,t) = \frac{\omega_n}{k_n h}\eta_n + \frac{\omega_m}{k_m h}\eta_m, \tag{7.4}$$

while the second-order velocity $U^{(2)}$ is given by (7.3 *a*) using $G_{\boldsymbol{U}}^\pm$ instead of $G_\eta^\pm$.

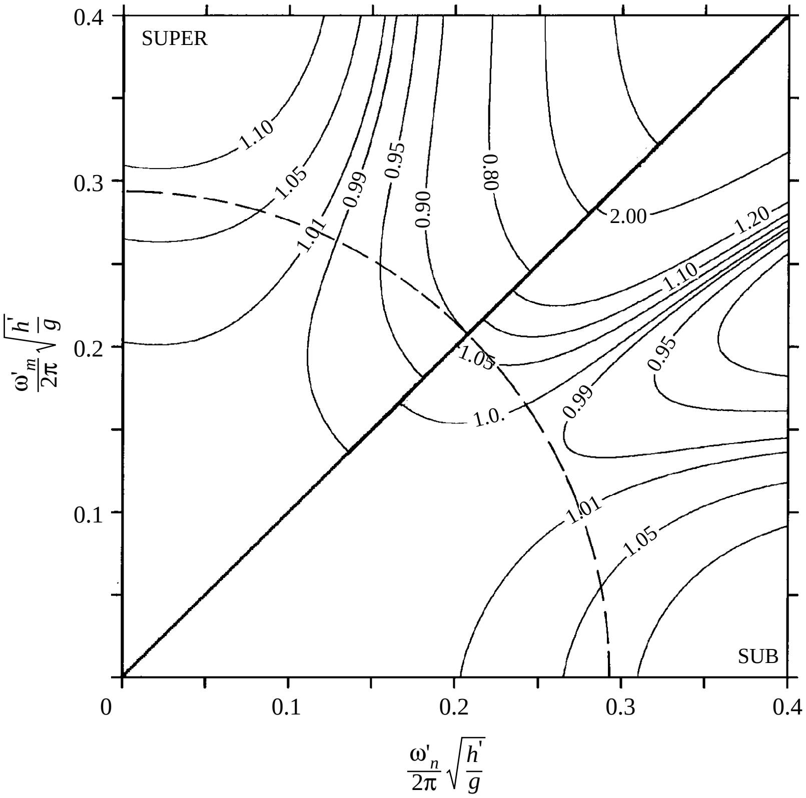

Figure 17. Ratio of second-order transfer functions, $G_\eta/G_\eta^{\rm Stokes}$, for $\boldsymbol{U}$ formulation. Boussinesq equations include (*a*) $O(\mu^4, \epsilon\mu^4)$, Padé [4,4]. Sub/superharmonic convention as in figure 15.

By substituting the combination of (7.1)–(7.4) into (4.4) and collecting terms of $O(\epsilon)$ we obtain the following algebraic system:

$$\begin{pmatrix} m_{11}^{(2)} & m_{12}^{(2)} \\ m_{21}^{(2)} & m_{22}^{(2)} \end{pmatrix} \begin{pmatrix} G_\eta^\pm \\ G_{\boldsymbol{U}}^\pm \end{pmatrix} = \begin{pmatrix} F_1^\pm \\ F_2^\pm \end{pmatrix}, \tag{7.5}$$

leading to the solution

$$G_\eta^\pm = \frac{F_1^\pm m_{22}^{(2)} - F_2^\pm m_{12}^{(2)}}{m_{11}^{(2)} m_{22}^{(2)} - m_{21}^{(2)} m_{12}^{(2)}}. \tag{7.6}$$

For the case of the bound superharmonic $\omega_p = \omega_n + \omega_m$, the coefficients in (7.6) read

$$m_{11}^{(2)} = \omega_p, \quad m_{12}^{(2)} = -k_p h, \tag{7.7 a}$$

$$m_{21}^{(2)} = -k_p(1 + \alpha_1\kappa_p^2 + \beta_1\kappa_p^4), \tag{7.7 b}$$

$$m_{22}^{(2)} = \omega_p(1 + (\alpha_1 + \tfrac{1}{3})\kappa_p^2 + (\beta_1 + \tfrac{1}{3}\alpha_1 - \tfrac{1}{45})\kappa_p^4) \tag{7.7 c}$$

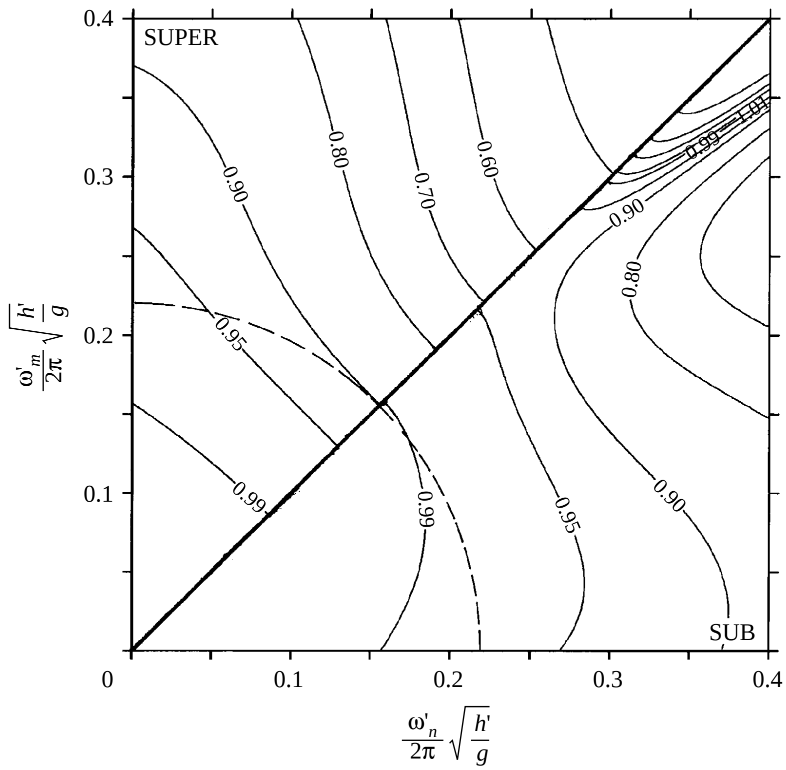

Figure 17. (*Cont.*) (*b*) $O(\mu^2, \epsilon\mu^2)$, Padé [2,2].

and

$$F_1^+ = \tfrac{1}{2}k_p\left(\frac{\omega_n}{k_n} + \frac{\omega_m}{k_m}\right), \tag{7.8 a}$$

$$\begin{aligned} F_2^+ = \tfrac{1}{2}k_p\frac{\omega_n\omega_m}{hk_nk_m}\bigg[1 &+ \tfrac{1}{3}\bigg((\kappa_n - \kappa_m)^2 + 3\alpha_1(\kappa_n + \kappa_m)^2 - \kappa_n^2\frac{\zeta_n}{\zeta_m}\left(\frac{2\kappa_n + 3\kappa_m}{\kappa_n + \kappa_m}\right) \\ &- \kappa_m^2\frac{\zeta_m}{\zeta_n}\left(\frac{2\kappa_m + 3\kappa_n}{\kappa_n + \kappa_m}\right)\bigg) + \tfrac{1}{45}\bigg(22\kappa_n^2\kappa_m^2 - (\kappa_n - \kappa_m)^4 \\ &+ \kappa_n^3\frac{\zeta_n}{\zeta_m}\left(\frac{4\kappa_n^2 + 10\kappa_n\kappa_m + 5\kappa_m^2}{\kappa_n + \kappa_m}\right) + \kappa_m^3\frac{\zeta_m}{\zeta_n}\left(\frac{4\kappa_m^2 + 10\kappa_n\kappa_m + 5\kappa_n^2}{\kappa_n + \kappa_m}\right)\bigg) \\ &+ \tfrac{1}{3}\alpha_1(\kappa_n + \kappa_m)\bigg(\kappa_n^3\left(1 - 2\frac{\zeta_n}{\zeta_m}\right) + \kappa_m^3\left(1 - 2\frac{\zeta_m}{\zeta_n}\right) \\ &- \kappa_n^2\kappa_m\left(1 + 3\frac{\zeta_n}{\zeta_m}\right) - \kappa_m^2\kappa_n\left(1 + 3\frac{\zeta_m}{\zeta_n}\right)\bigg) + \beta_1(\kappa_n + \kappa_m)^4\bigg], \end{aligned} \tag{7.8 b}$$

where ζ is defined by (4.19), κ is defined by (3.12) and where $k_p = k_n + k_m$ and $\kappa_p = \kappa_n + \kappa_m$. The subharmonic transfer function is obtained simply by changing the sign of the quantities $(\omega_m, k_m, \kappa_m)$ in (7.7) and (7.8). For the case of $\omega_n = \omega_m$, the problem simplifies to the second-harmonic transfer analysed in § 4 *c*, and (7.7) reduce to (4.7 *a*) while (7.8) reduce to (4.7 *b*), except for a factor 2 which is cancelled by $\delta = \frac{1}{2}$ in (7.3 *a*).

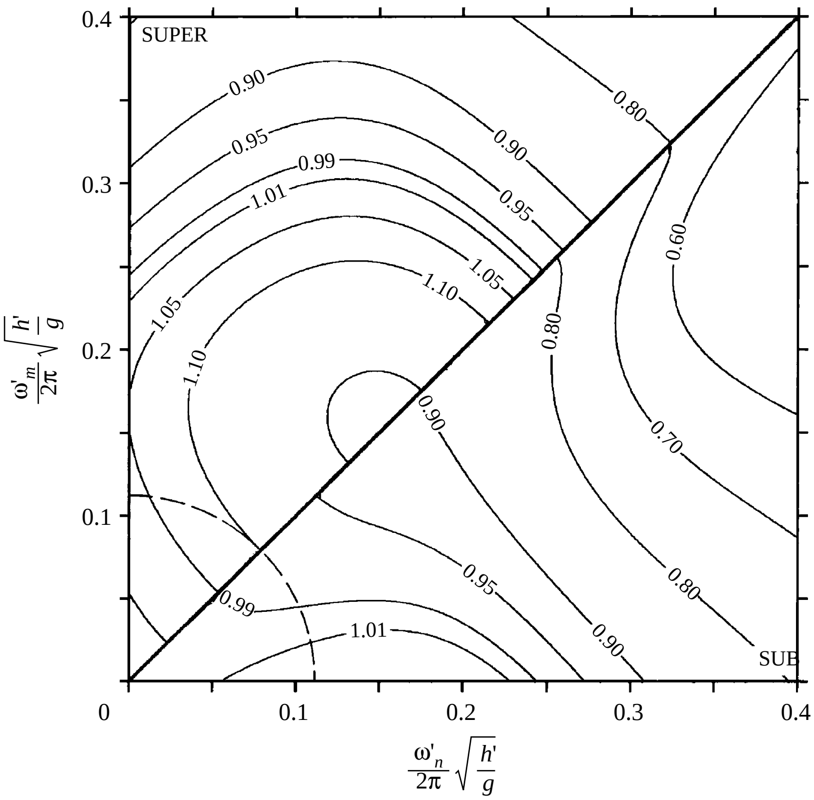

Figure 17. (*Cont.*) (*c*) $O(\mu^2, \epsilon)$, Padé [2,2].

Figure 17*a* shows the ratio between the Boussinesq transfer function (7.6) and the target transfer function determined from the Laplace equation (shown in figure 15). Again, the upper triangle represents the superharmonic transfer from ω_n and ω_m to $\omega_p = \omega_n + \omega_m$, while the lower triangle represents the subharmonic transfer from ω_n and ω_m to $\omega_p = \omega_n - \omega_m$. The figure also includes a 10% error circle (as defined in § 7 *a*) for which the radius is $\Omega_{\mathrm{R}} = 0.29$ corresponding to $h'/L'_{0,\mathrm{R}} = 0.53$, i.e. slightly exceeding the traditional deep water limit.

If we neglect terms of $O(\mu^4, \epsilon\mu^4)$ and retain $O(\mu^2, \epsilon\mu^2)$ terms in the governing equations, the result will be a set of Serre-type equations (Serre 1953) enhanced to incorporate Padé [2,2] dispersion characteristics. The resulting transfer function errors are shown in figure 17*b* and we notice that the radius of the 10% error circle is reduced to $\Omega_{\mathrm{R}} = 0.22$ corresponding to $h'/L'_0 = 0.30$. The traditional Boussinesq equations enhanced to Padé [2,2] dispersion characteristics correspond to a further neglect of $O(\epsilon\mu^2)$ terms and this case is shown in figure 17*c*. The radius of the 10% error circle is reduced to $\Omega_{\mathrm{R}} = 0.11$, i.e. $h'/L'_{0,\mathrm{R}}$ as low as 0.08.

Finally, for reference, figure 18 shows the transfer function error for the lower-order Boussinesq equations analysed by Madsen & Sørensen (1993). These equations are equivalent to the equations behind figure 17*c* except for using the flux (depth-integrated velocity) instead of the depth-averaged velocity as dependent variable, and the values of $\Omega_{\mathrm{R}} = 0.10$ and $h'/L'_{0,\mathrm{R}} = 0.06$ are similar to those of figure 17*c*. However, it is interesting to note the qualitative difference for the superharmonic

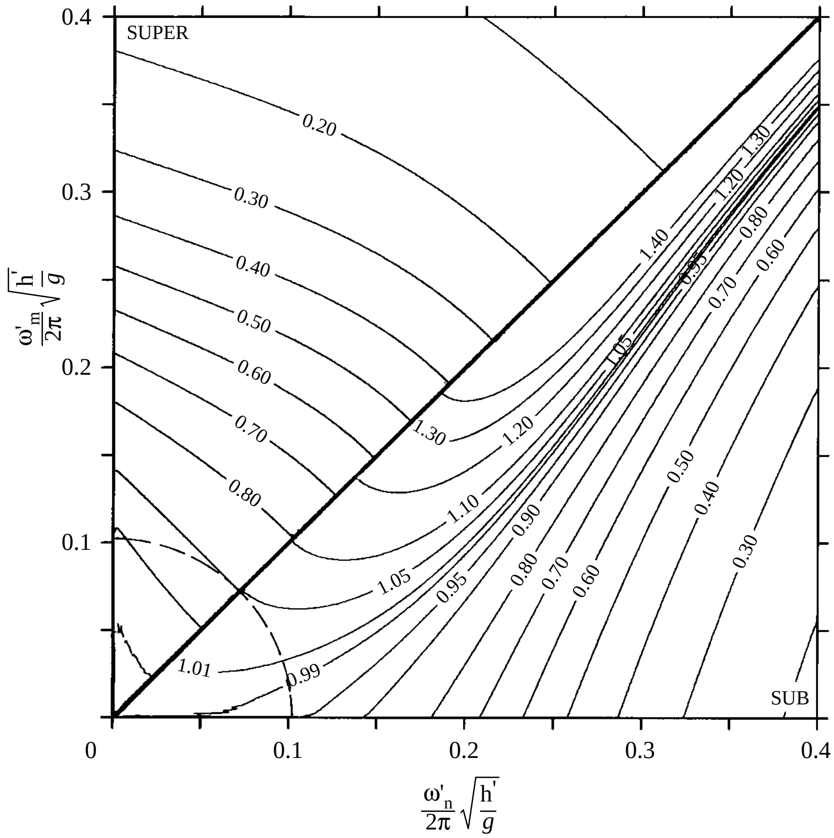

Figure 18. Ratio of second-order transfer functions, $G_\eta/G_\eta^{\mathrm{Stokes}}$, for flux formulation (depth-integrated velocity (Madsen & Sørensen 1993)).

transfer which is too large for the depth-averaged velocity formulation (figure 17*c*) while it is too small for the depth-integrated one (figure 18). For shoaling waves this difference shows up as too peaky wave profiles when using the depth-averaged velocity formulation and the opposite when using the equations in terms of the depth-integrated one. Table 5 summarizes the results for the 10% error limits.

(*c*) *Analysis of $\tilde{u}$ formulation from § 6*

In this section we shall derive the $G_\eta^\pm$ transfer function on the basis of the enhanced $\tilde{u}$ formulation of the Boussinesq equations given in (6.3). In one dimension and on a horizontal bottom, these equations simplify to (6.5), which will be used as a starting point for the derivation.

Again, (7.1) is a solution to the linearized equations, which implies that (ω_n, k_n) and (ω_m, k_m) satisfy the dispersion relation (6.8). Furthermore, the first-order velocity, $\tilde{u}^{(1)}$ can be expressed by

$$\tilde{u}^{(1)}(x,t) = \frac{\gamma_n\omega_n}{k_n h}\eta_n + \frac{\gamma_m\omega_m}{k_m h}\eta_m, \tag{7.9}$$

with (γ_n, γ_m) satisfying (6.7), while the second-order velocity $\tilde{u}^{(2)}$ is given by using $G_{\tilde{\boldsymbol{u}}}^\pm$ instead of $G_\eta^\pm$ in (7.3 *a*). By substituting the combination of (7.1)–(7.3) into (6.5)

Table 5. *Maximum frequency or depth for a 10% error limit on* $G_\eta^\pm$

(For $\sqrt{\Omega_n^2+\Omega_m^2} \leqslant \Omega_{\mathrm{R}}$, the error is less than 10%. The last case is for a depth-integrated velocity formulation.)

equations in $\boldsymbol{U}$	Padé	Ω_{R}	$h'/L'_{0,\mathrm{R}}$	figure
$(\mu^4, \epsilon\mu^4)$	[4,4]	0.29	0.53	17*a*
$(\mu^2, \epsilon\mu^2)$	[2,2]	0.22	0.30	17*b*
(μ^2, ϵ)	[2,2]	0.11	0.08	17*c*
(μ^2, ϵ) (flux)	[2,2]	0.10	0.06	18

and collecting terms of $O(\epsilon)$, we once again obtain the algebraic system given in (7.5) and the solution given in (7.6).

For the case of the bound superharmonic $\omega_p = \omega_n + \omega_m$, the coefficients in (7.5) read

$$\left.\begin{aligned} m_{11}^{(2)} &= \omega_p(1+\beta_1\kappa_p^2), \quad m_{12}^{(2)} = -k_p h(1-(\alpha+\tfrac{1}{3}-\beta_1)\kappa_p^2), \\ m_{21}^{(2)} &= -k_p(1+\alpha_1\kappa_p^2), \quad m_{22}^{(2)} = \omega_p(1-(\alpha-\alpha_1)\kappa_p^2) \end{aligned}\right\} \tag{7.10}$$

and

$$F_1^+ = \tfrac{1}{2}k_p\left(\frac{\gamma_n\omega_n}{k_n}+\frac{\gamma_m\omega_m}{k_m}\right)\left(1+\beta_1(\kappa_n+\kappa_m)^2-\alpha\left(\frac{\gamma_n\zeta_n\kappa_n^2+\gamma_m\zeta_m\kappa_m^2}{\gamma_n\zeta_n+\gamma_m\zeta_m}\right)\right), \tag{7.11 a}$$

$$F_2^+ = \tfrac{1}{2}k_p\frac{\gamma_n\gamma_m\omega_n\omega_m}{hk_nk_m} \times\left(1+\alpha_1(\kappa_n+\kappa_m)^2-\alpha(\kappa_n^2+\kappa_m^2)-\kappa_n\kappa_m-\kappa_n^2\frac{\zeta_n}{\gamma_m\zeta_m}-\kappa_m^2\frac{\zeta_m}{\gamma_n\zeta_n}\right), \tag{7.11 b}$$

where ζ is defined by (4.19), κ is defined by (3.12) and where $k_p = k_n + k_m$ and $\kappa_p = \kappa_n + \kappa_m$. Again the subharmonic transfer function is obtained by changing the sign of the quantities $(\omega_m, k_m, \kappa_m)$ in (7.10) and (7.11). For the case of $\omega_n = \omega_m$ the problem simplifies to the second-harmonic transfer analysed in § 6 *b*, and (7.10) reduce to (6.10 *a*) while (7.11) reduce to (6.10 *b*), except for a factor 2 which is cancelled by $\delta = \frac{1}{2}$ in (7.3 *a*).

Figure 19*a* shows the ratio of $G_\eta/G_\eta^{\mathrm{Laplace}}$, where the Boussinesq transfer function has been determined on the basis of the coefficient set I given by (6.9 *a*). Again the upper triangle in the figure represents the superharmonic transfer from ω_n and ω_m to $\omega_p = \omega_n + \omega_m$, while the lower triangle represents the subharmonic transfer from ω_n and ω_m to $\omega_p = \omega_n - \omega_m$. Equivalent results are shown in figure 19*b* neglecting the dispersion enhancement (now using $\alpha = -\frac{2}{5}$ giving dispersion corresponding to Padé [2,2] instead of Padé [4,4]) and in figure 19*c* further neglecting $O(\epsilon\mu^2)$ terms, i.e. retaining only terms of $O(\mu^2, \epsilon)$. The three respective sets of equations result in 10% error circles characterized by $\Omega_{\mathrm{R}} = (0.23, 0.20, 0.12)$ or $h'/L'_{0,\mathrm{R}} = (0.32, 0.25, 0.09)$. Comparing the first two results (figure 19*a*,*b*) it appears that the enhancement, which was introduced with the purpose of improving dispersion, also has a positive effect on the accuracy of nonlinearity. Comparison between the last two results (figure 19*b*,*c*)

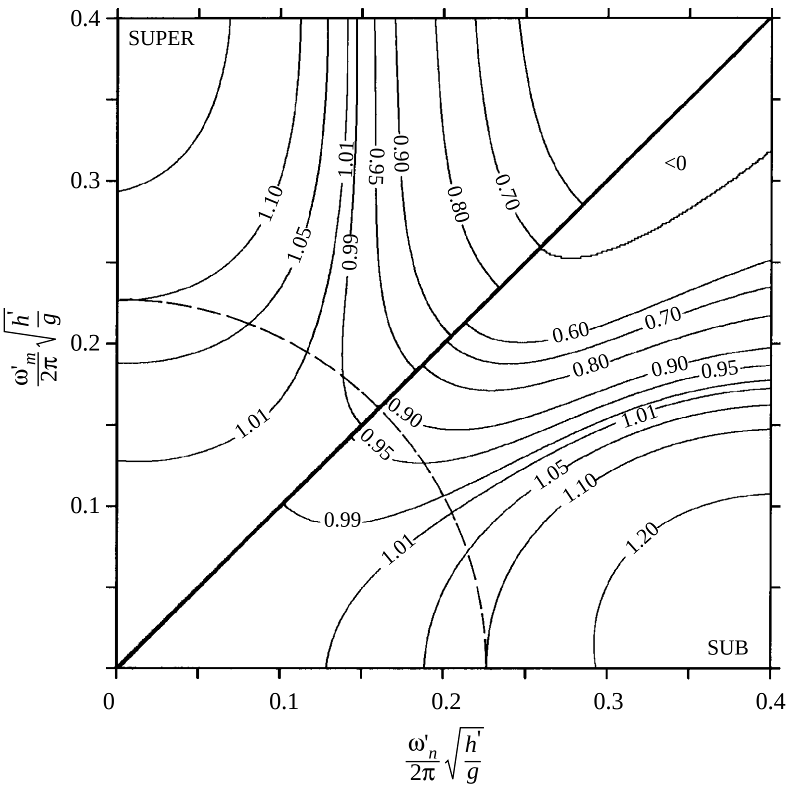

Figure 19. Ratio of second-order transfer function, $G_\eta/G_\eta^{\mathrm{Stokes}}$ for $\tilde{\boldsymbol{u}}$ formulation. Boussinesq equations include (*a*) $O(\mu^2, \epsilon\mu^2)$, Padé [4,4].

shows the expected significant improvement of second-order transfer obtained by including extra nonlinear terms in the equations.

Table 6 summarizes the results for the 10% error limits. The equations behind figure 19*b*,*c* were first derived by Wei *et al*. (1995) and Nwogu (1993).

(*d*) *Comparison between* $\boldsymbol{U}$ *and* $\tilde{\boldsymbol{u}}$ *formulations*

This section is closed with a discussion on the accuracy of second-order transfer for the two different velocity formulations. First of all, it is noted that for the $\boldsymbol{U}$ formulation equations were derived retaining $O(\mu^4, \epsilon\mu^4)$, while the $\tilde{\boldsymbol{u}}$ equations only included $O(\mu^2, \epsilon\mu^2)$ terms. Thus the best of the $\tilde{\boldsymbol{u}}$ equations (figure 19*a*) can be expected to give results of similar accuracy as for the $O(\mu^2, \epsilon\mu^2)$ $\boldsymbol{U}$ formulation (figure 17*b*) as confirmed by their similar 10% error limits ($h'/L'_{0,\mathrm{R}} = 0.32$ and 0.30, respectively).

However, a closer inspection of figures 19*a* and 17*b* reveal that outside the 10% error circle figure 19*a* is actually superior to figure 17*b* in large regions. The explanation for this lies in the dispersion relation which also effects the results for second-order transfer. While the best $\tilde{\boldsymbol{u}}$ formulation has Padé [4,4] dispersion, the $O(\mu^2, \epsilon\mu^2)$ $\boldsymbol{U}$ formulation only corresponds to Padé [2,2]. Dropping the dispersion enhancement for

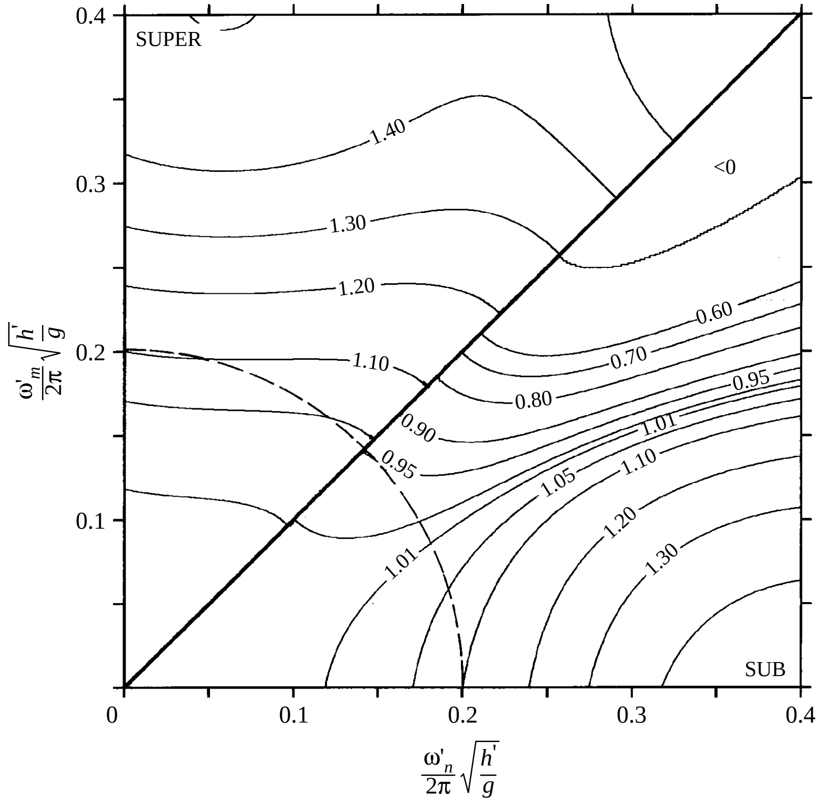

Figure 19. (*Cont.*) (*b*) $O(\mu^2, \epsilon\mu^2)$, Padé [2,2].

the $\tilde{\boldsymbol{u}}$ formulation (figure 19*b*) the results are no longer as good as for the equivalent order $\boldsymbol{U}$ formulation (figure 17*b*).

Finally, a comparison between results of the $O(\mu^2, \epsilon)$ equations from the two velocity formulations in question (figures 17*c* and 19*c*) shows that while the $\tilde{\boldsymbol{u}}$ formulation has slightly better 10% error limit, the $\boldsymbol{U}$ formulation is favoured if a wider frequency range is considered.

8. Wave–current interaction, Doppler shift and wave blocking

(*a*) *Introduction*

In this section we shall focus on wave–current interaction in the framework of Boussinesq equations. It is well known that one consequence of the nonlinearity of the Boussinesq equations is the automatic inclusion of wave-averaged effects such as radiation stress, setup, undertow and wave-induced currents. This is, however, not a guarentee for a correct representation of, for example, the Doppler shift in connection with current refraction, and, in fact, most Boussinesq-type equations fail to model this phenomenon accurately.

In the literature we find only few examples, where the problem of wave–current interaction has been treated explicitly in the framework of Boussinesq-type equations (Yoon & Liu 1989; Prüser & Zielke 1990; Kristensen 1995; Chen *et al*. 1998). In

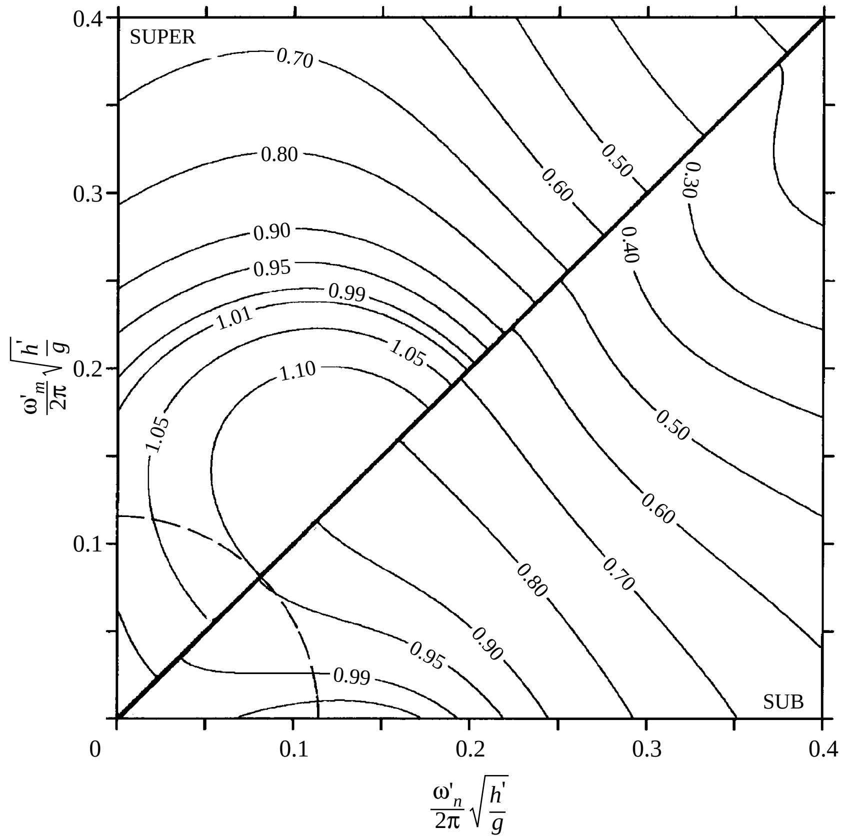

Figure 19. (*Cont.*) (*c*) $O(\mu^2, \epsilon)$, Padé [2,2].

Table 6. *Maximum frequency or depth for a 10% error limit on* $G_\eta^\pm$
(For $\sqrt{\Omega_n^2 + \Omega_m^2} \leqslant \Omega_R$, the error is less than 10%.)

equations in $\tilde{\boldsymbol{u}}$	Padé	Ω_R	$h'/L'_{0,R}$	figure
$(\mu^2, \epsilon\mu^2)$	[4,4]	0.23	0.32	19*a*
$(\mu^2, \epsilon\mu^2)$	[2,2]	0.20	0.25	19*b*
(μ^2, ϵ)	[2,2]	0.09	0.09	19*c*

the classical Boussinesq equations where $\epsilon = O(\mu^2)$, the presence of an ambient current calls for special attention and scaling, as shown by Yoon & Liu (1989). They considered the interaction of weakly nonlinear shallow water waves with slowly varying currents and topography, and assumed the magnitude of the current velocity to be stronger than that of the characteristic wave particle velocity, but weaker than that of the wave group velocity. Furthermore, the horizontal length scales of the current variation and of the depth variation were assumed to be longer than the characteristic wavelength. In comparison to the classical Boussinesq equations of Peregrine (1967), the equations derived by Yoon & Liu and by Prüser & Zielke had additional terms, which were proportional to the product of the current velocity (U'^{C})

and third derivatives of the wave velocity (U'^{W}). It turns out that these additional terms are contained in the $\epsilon\mu^2$ terms given by, for example, (3.8 *b*) or (4.3 *b*).

In the present work, we have avoided the complication of separate scalings of waves and currents by taking ϵ as arbitrary. Consequently, as this section will demonstrate, the equations of this work will incorporate the same linear dispersion characteristics for the relative wave motion with or without ambient currents. The analysis made in the following will be restricted to one horizontal dimension, and we shall consider the velocity variable U' as consisting of two parts, a wave orbital velocity U'^{W} and a current velocity U'^{C}. The current velocity is assumed to be uniform over depth and allowed to be as strong as the phase celerity of the wave. Hence a natural scaling of the current is

$$U^{\mathrm{C}} = \frac{U'^{\mathrm{C}}}{\sqrt{gh'_0}} = O(1), \tag{8.1 a}$$

while the wave particle velocity is scaled according to (2.1), i.e.

$$U^{\mathrm{W}} = \frac{U'^{\mathrm{W}}}{\epsilon\sqrt{gh'_0}} = O(1). \tag{8.1 b}$$

Also the total velocity U is scaled according to (2.1), i.e. as (8.1 *b*), and consequently the combination of (8.1 *a*) and (8.1 *b*) yields the following relation in non-dimensional variables:

$$U = U^{\mathrm{W}} + \frac{U^{\mathrm{C}}}{\epsilon} = O\left(\frac{1}{\epsilon}\right). \tag{8.1 c}$$

The temporal variation of the current will typically be orders of magnitude slower than that of the waves, while the spatial variation is closely related to the variation of the bottom bathymetry. We shall assume that the current will vary on a larger spatial scale than the wavelength scale, and that this will compensate for the strength of the current in such a way that

$$hU^{\mathrm{C}}_x = O(hU^{\mathrm{W}}_x) = O(1), \tag{8.2}$$

while higher derivatives of U^{C} will be smaller than similar derivatives of U^{W}.

Hence the combination of (8.1 *c*) and (8.2) implies the following re-ordering of terms in the Boussinesq equations derived in the previous sections: terms containing factors of U rather than derivatives of U will move up in the hierarchy; for example, from $O(\epsilon)$ to $O(1)$, from $O(\epsilon\mu^2)$ to $O(\mu^2)$ and from $O(\epsilon\mu^4)$ to $O(\mu^4)$. This explains why the additional current-terms derived by Yoon & Liu in the lower-order Boussinesq equations are contained in the $\epsilon\mu^2$ terms given in §§ 3 and 4.

The spatial variation of the current is generally related to the bathymetric changes and by continuity, we have

$$hU^{\mathrm{C}}_x = O(U^{\mathrm{C}}h_x), \tag{8.3 a}$$

which in combination with (8.2) yields

$$U^{\mathrm{C}}h_x = O(1). \tag{8.3 b}$$

This means that strong currents can be treated only in connection with weakly varying bathymetries, while weak currents do not imply any restrictions on the bathymetric variations.

(*b*) *Analysis of U formulation from* § 4

In this section we shall demonstrate that the new enhanced higher-order Boussinesq equations (4.2) and (3.2) formulated in the depth-averaged velocity U incorporate excellent characteristics with respect to wave–current interaction. In the analysis we assume a constant depth and a constant current and on the basis of (8.2) the one-dimensional versions of (4.2) and (3.2) simplify to

$$\eta_t + hU_x^{\mathrm{W}} + U^{\mathrm{C}}\eta_x = (\epsilon) \tag{8.4 a}$$

and

$$\begin{aligned} &U_t^{\mathrm{W}} + \eta_x + \mu^2(-\alpha_1 h^2 \eta_{xxx} - (\alpha_1 + \tfrac{1}{3})h^2 U_{xxt}^{\mathrm{W}}) \\ &\quad + \mu^4(\beta_1 h^4 \eta_{xxxxx} + (\beta_1 + \tfrac{1}{3}\alpha_1 - \tfrac{1}{45})h^4 U_{xxxxt}^{\mathrm{W}}) \\ &\quad + U^{\mathrm{C}}[U_x^{\mathrm{W}} - \mu^2(\alpha_1 + \tfrac{1}{3})h^2 U_{xxx}^{\mathrm{W}} + \mu^4(\beta_1 + \tfrac{1}{3}\alpha_1 - \tfrac{1}{45})h^4 U_{xxxxx}^{\mathrm{W}}] = O(\epsilon). \end{aligned} \tag{8.4 b}$$

Again we look for wave solutions of the form

$$\eta = a_1 \cos\theta, \quad U^{\mathrm{W}} = U_1 \cos\theta, \tag{8.5}$$

and by inserting (8.5) in (8.4) we obtain the algebraic system of (3.13) with coefficients defined by

$$\left.\begin{aligned} m_{11}^{(1)} = \omega - kU^{\mathrm{C}}, \quad m_{12}^{(1)} = -kh, \quad m_{21}^{(1)} = -k(1 + \alpha_1\kappa^2 + \beta_1\kappa^4), \\ m_{22}^{(1)} = (\omega - kU^{\mathrm{C}})(1 + (\alpha_1 + \tfrac{1}{3})\kappa^2 + (\beta_1 + \tfrac{1}{3}\alpha_1 - \tfrac{1}{45})\kappa^4). \end{aligned}\right\} \tag{8.6}$$

The resulting dispersion relation reads

$$\frac{(\omega - kU^{\mathrm{C}})^2}{k^2 h} = \frac{1 + \alpha_1\kappa^2 + \beta_1\kappa^4}{1 + (\alpha_1 + \frac{1}{3})\kappa^2 + (\beta_1 + \frac{1}{3}\alpha_1 - \frac{1}{45})\kappa^4}. \tag{8.7}$$

Obviously, this provides the correct form of the Doppler shift. Padé [4,4] dispersion characteristics now apply in connection with the relative phase celerity, provided that $\alpha_1 = \frac{1}{9}$ and $\beta_1 = \frac{1}{945}$ is chosen. If we neglect the μ^4 terms in (8.4 *b*) corresponding to a lower-order enhanced set of equations and use $\alpha_1 = \frac{1}{15}$, the resulting dispersion relation (8.7) simplifies to a Padé [2,2] expansion. This result was previously obtained by Kristensen (1995). A discussion of (8.7) will be given in § 8 *d*.

(*c*) *Analysis of ũ formulation from* § 6

In this section we shall demonstrate that also the enhanced $\tilde{u}$ formulation derived in § 6 incorporates excellent characteristics with respect to wave–current interaction. Again we assume a constant depth and a constant current and (8.1 *c*) is replaced by

$$\tilde{u} = \tilde{u}^{\mathrm{W}} + \frac{U^{\mathrm{C}}}{\epsilon} = O\left(\frac{1}{\epsilon}\right). \tag{8.8}$$

Hence, by the use of (8.8), we simplify (6.5) to

$$\eta_t + h\tilde{u}_x^{\mathrm{W}} + \mu^2((\alpha + \tfrac{1}{3} - \beta_1)h^3\tilde{u}_{xxx}^{\mathrm{W}} - \beta_1 h^2 \eta_{xxt}) + U^{\mathrm{C}}(\eta_x - \mu^2 h^2 \beta_1 \eta_{xxx}) = O(\epsilon) \tag{8.9 a}$$

and

$$\tilde{u}_t^{\mathrm{W}} + \eta_x + \mu^2((\alpha - \alpha_1)h^2\tilde{u}_{xxt}^{\mathrm{W}} - \alpha_1 h^2 \eta_{xxx}) + U^{\mathrm{C}}(\tilde{u}_x^{\mathrm{W}} + \mu^2 h^2(\alpha - \alpha_1)\tilde{u}_{xxx}^{\mathrm{W}}) = O(\epsilon). \tag{8.9 b}$$

Again, we look for wave solutions of the form (8.5) and this leads to the algebraic system of (3.13) with coefficients defined by

$$\left.\begin{aligned} m_{11}^{(1)} &= (\omega - kU^{\mathrm{C}})(1+\beta_1\kappa^2), \quad m_{12}^{(1)} = -kh(1-(\alpha+\tfrac{1}{3}-\beta_1)\kappa^2), \\ m_{21}^{(1)} &= -k(1+\alpha_1\kappa^2), \quad m_{22}^{(1)} = (\omega - kU^{\mathrm{C}})(1-(\alpha-\alpha_1)\kappa^2). \end{aligned}\right\} \tag{8.10}$$

The resulting dispersion relation reads

$$\frac{(\omega - kU^{\mathrm{C}})^2}{k^2 h} = \frac{1+(\alpha_1+\beta_1-\alpha-\frac{1}{3})\kappa^2 + \alpha_1(\beta_1-\alpha-\frac{1}{3})\kappa^4}{1+(\alpha_1+\beta_1-\alpha)\kappa^2+\beta_1(\alpha_1-\alpha)\kappa^4}, \tag{8.11}$$

and $(\alpha, \beta_1, \alpha_1)$ can be chosen from the sets defined by (6.9). Hence, again we have obtained a correct Doppler shift with Padé [4,4] dispersion characteristics for the intrinsic wave motion. This result is also reported in Chen *et al*. (1996), where the derivation procedure follows Yoon & Liu (1989) except for the choice of velocity variable. A further discussion of (8.11) will be given in § 8 *d*.

(*d*) *Doppler shift and wave blocking*

In the previous two sections we have demonstrated that a correct Doppler shift combined with a phase celerity as accurate as a Padé [4,4] expansion of Stokes's linear celerity can be achieved on the basis of two different sets of enhanced Boussinesq equations: the U formulation from § 4 retaining terms of $O(\mu^4, \epsilon\mu^4)$, and the $\tilde{u}$ formulation from § 6 retaining terms of $O(\mu^2, \epsilon\mu^2)$.

For comparison, a linear analysis of the equations of Yoon & Liu (1989) leads to

$$\frac{(\omega' - k'U'^{\mathrm{C}})^2}{gh'k'^2} = \frac{1}{1+\frac{1}{3}(k'h')^2}, \tag{8.12}$$

while the reference solution according to Stokes's linear theory reads

$$\frac{(\omega' - k'U'^{\mathrm{C}})^2}{gh'k'^2} = \frac{\tanh(k'h')}{k'h'}, \tag{8.13}$$

where we have returned to dimensional variables.

In the following we shall concentrate on the case of opposing currents and solve the dispersion relations with respect to the wavenumber, $k'h'$ as a function of the Froude number, F_c and h'/L_0' defined by

$$F_c \equiv \frac{U'^{\mathrm{C}}}{\sqrt{gh'}}, \quad \frac{h'}{L_0'} \equiv \frac{\omega'^2 h'}{2\pi g}, \tag{8.14}$$

where L_0' is the linear deep-water wavelength in the absense of the current.

Figure 20*a* shows the reference solution of (8.13) and we notice how wavenumbers (for a given h'/L_0') increase for increasing absolute values of F_c and how wave blocking prevents solutions for certain combinations of F_c and h'/L_0'. The blocking generally implies that the current speed exceeds the group velocity of the wave: for small values of h'/L_0' this requires that F_c approaches -1, corresponding to critical flow, while waves with larger values of h'/L_0' will be blocked for much smaller values of F_c.

Figure 20*b* shows the Padé [4,4] solution obtained from the two new sets of equations, i.e. from either (8.7) or (8.11). In general, the agreement with the target in figure 20*a* is excellent and the isolines representing, for example, $k'h' = 1$, 2, 3 and

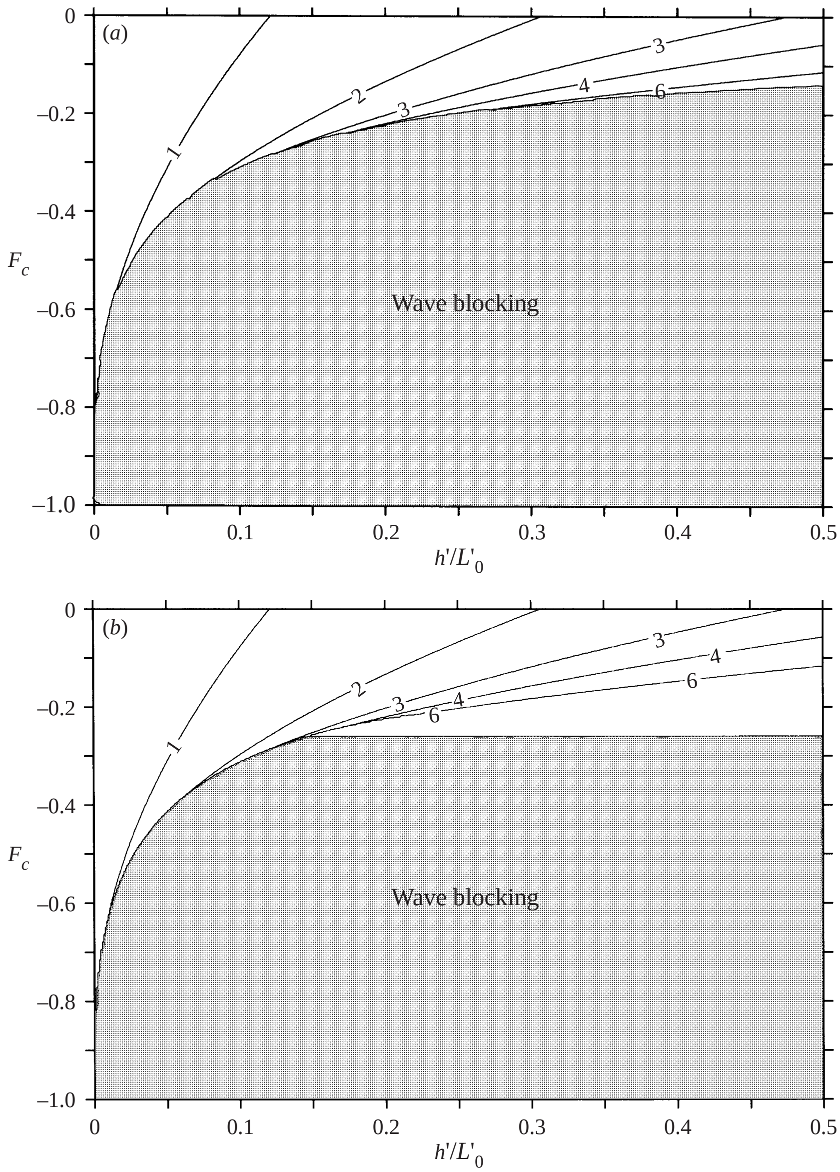

Figure 20. Wavenumber solutions, $k'h'$ in opposing currents as a function of F_c and h'/L'_0 as defined by (8.14). (*a*) Stokes's linear theory, i.e. (8.13); (*b*) Padé [4,4] obtained with new equations, i.e. (8.7) or (8.11).

4 are almost exact, while only minor errors occur for $k'h'$ up to 6. The accuracy is quantified in figure 21, which shows the tracks in $(F_c, h'/L'_0)$-space of 2% wavenumber errors, $(k - k^{\mathrm{Stokes}})/k^{\mathrm{Stokes}}$ obtained from various Boussinesq-type approximations (dotted lines) and the blocking curve of Stokes (full line). This is a measure of the range of validity of each of the approximations and we notice that the Padé [4,4] is clearly superior to the Padé [2,2] (as obtained by Kristensen (1995)), which again is superior to the Padé [0,2] as formulated by Yoon & Liu. Hence obviously the range of

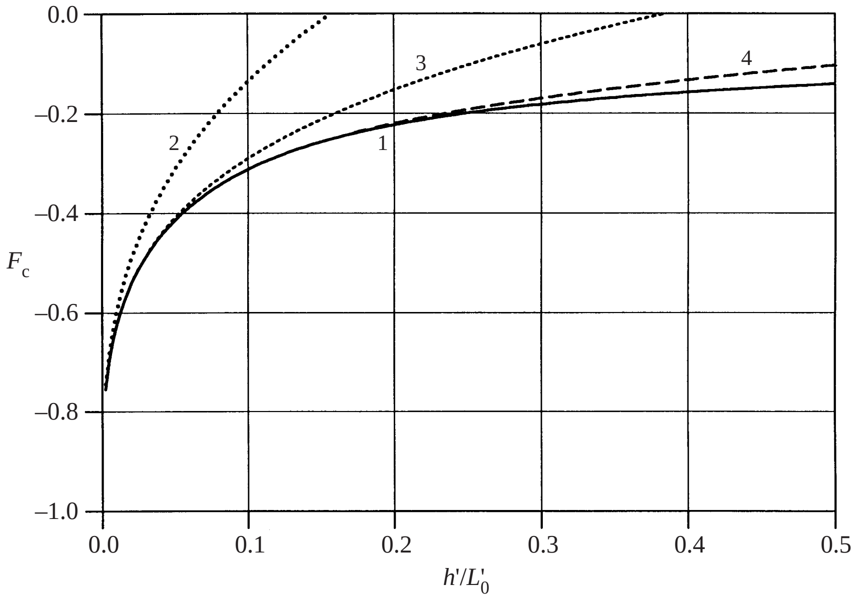

Figure 21. Tracks of 2% wavenumber errors, $(k - k^{\text{Stokes}})/k^{\text{Stokes}}$. (2) Yoon & Liu (1989); (3) Padé [2,2]; (4) Padé [4,4]; (1) Blocking curve according to Stokes's theory.

applicability of the new equations is significantly improved in comparison to previous Boussinesq-type formulations.

Figure 22 shows the classical wave blocking curves, $F_c = -c_g/\sqrt{gh}$, where c_g is the relative group velocity. We notice that the solution for Padé [0,2] dispersion given by Yoon & Liu qualitatively retains the blocking property from the target solution for any negative Froude number, although blocking occurs for much too weak a current. Conversely, the blocking curves marked 3 and 4 corresponding to Padé [2,2] and Padé [4,4] dispersion are seen to terminate at $(F_c, h'/L'_0) = (-0.34, 0.087)$ and $(-0.21, 0.225)$, respectively. However, these blocking curves branch off as $k'h'$ goes to infinity when F_c goes to $-1/\sqrt{15} \approx -0.258$ (for Padé [4,4]) and $-1/\sqrt{6} \approx -0.408$ (for Padé [2,2]). Since dispersion vanishes in the limit, the relative phase celerity c equals c_g and we have $F_c = -c_g/\sqrt{gh} = -c/\sqrt{gh}$. Figure 23 shows the wavenumbers determined along the blocking curves in figure 22. We notice that the wavenumbers obtained by Padé [4,4] and Padé [2,2] are in good agreement with the target solution up to $h'/L'_0 \approx 0.20$ and 0.08, respectively, while the wavenumbers obtained from the formulation of Yoon & Liu are generally much too small.

The reason why wave blocking does not occur for short waves or weak currents in case of Padé [2,2] or Padé [4,4] dispersion characteristics can be seen from figure 24, which shows $F_c = -c'_g/\sqrt{gh'}$ versus $k'h'$. We notice that for the target solution of Stokes (and for Yoon & Liu) this goes to zero for large values of $k'h'$, indicating that no matter how weak the current is, blocking will occur provided the waves are sufficiently short. Hence as $k'h'$ runs through the interval $[0; \infty]$, the blocking value of F_c runs through $[-1; 0]$. However, for the case of Padé [2,2] and Padé [4,4] this F_c interval is seen to be limited to $[-1; -0.34]$ and $[-1; -0.21]$, respectively, and the limits of these intervals correspond to the cut-off values of the blocking curves in figure 22.

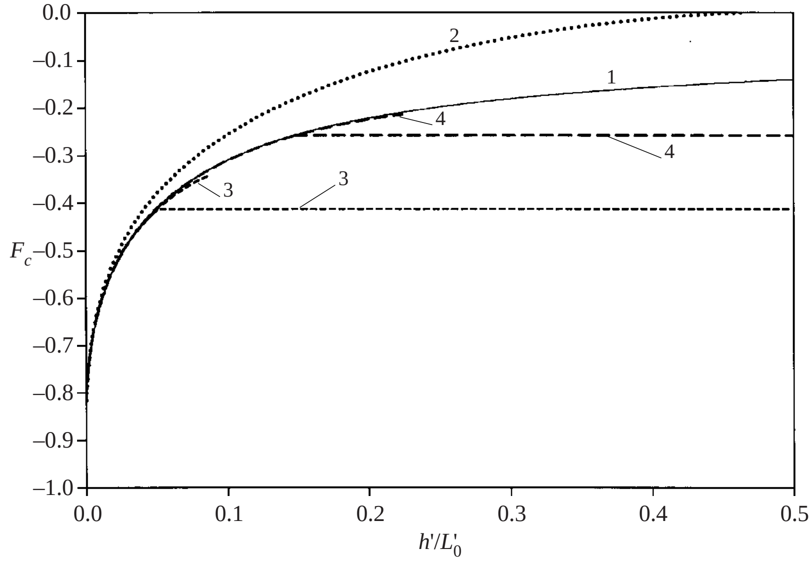

Figure 22. Wave blocking curve in $(F_c, h'/L'_0)$ space. (1) Target solution of Stokes; (2) Yoon & Liu (1989), Padé [0,2]; (3) Padé [2,2]; (4) Padé [4,4].

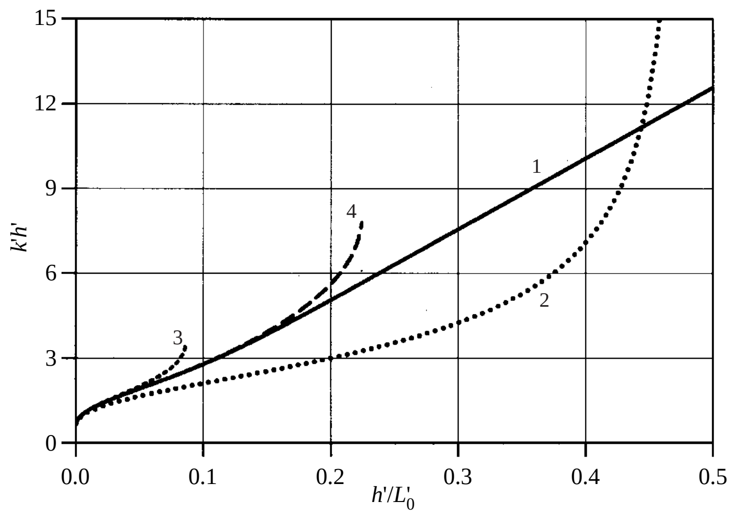

Figure 23. Wavenumber solution along the blocking curve. (1) Target solution of Stokes; (2) Yoon & Liu (1989), Padé [0,2]; (3) Padé [2,2]; (4) Padé [4,4].

9. Summary and conclusions

Boussinesq-type equations of higher order in dispersion as well as in nonlinearity are derived for waves (and wave–current interaction) over an uneven bottom. Formulations are given in terms of various velocity variables such as the depth-averaged velocity and the particle velocity at the still water level and at an arbitrary vertical location. The equations are enhanced and analysed with emphasis on linear dispersion and shoaling characteristics and nonlinear properties for large wavenumbers.

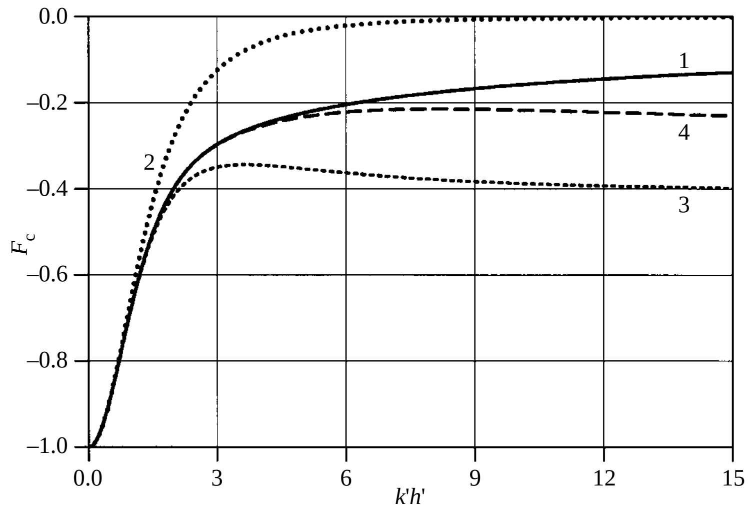

Figure 24. Froude number F_c determined as minus the relative group velocity $c'_{\mathrm{g}}/\sqrt{gh'}$ versus $k'h'$. (1) Target solution of Stokes; (2) Yoon & Liu (1989), Padé [0,2]; (3) Padé [2,2]; (4) Padé [4,4].

The starting point for the derivation is the Laplace equation with nonlinear boundary conditions expressed in terms of the velocity potential. Important scaling parameters are μ and ϵ, representing linear dispersion and nonlinearity, respectively. We assume μ to be small and take ϵ to be arbitrary initially. As the first step in the development of the new equations, we expand the velocity potential as a power series in the vertical coordinate. By the use of the governing equations the resulting series expansion is determined in terms of the horizontal and vertical velocity components at the still water level (SWL) and in powers of the dispersion parameter, μ. Truncating at $O(\mu^6)$, we derive higher-order Boussinesq-type equations in terms of the horizontal velocity at the SWL retaining all orders of nonlinearity (§ 2).

These equations are then recast in terms of the depth-averaged velocity, $\boldsymbol{U}$ (§ 3). For simplicity we leave out the $O(\epsilon^2\mu^4)$ terms, which corresponds to an assumption of $\epsilon = O(\mu)$. The resulting equations contain fifth-derivative terms in the momentum equation, and first-derivative terms in the continuity equation. Unfortunately, it turns out that the linear dispersion relation embedded in the equations contains a singularity at $k'h' \approx 4.2$. This singularity spills over to the phase mismatch with higher harmonics and, for example, the second harmonic contains a singularity at $k'h' = \sqrt{3}$, while the third harmonic contains two singularities at $k'h' = \sqrt{3}/\sqrt{2}$ and $k'h' = \sqrt{3}$. This makes these equations useless for practical purposes, since instabilities will occur in connection with numerical time integration.

In § 4 the higher-order equations from § 3 are enhanced by borrowing from terms belonging to orders above the ones retained. This technique removes the singularities and results in linear dispersion characteristics corresponding to a Padé [4,4] expansion in $k'h'$ of the squared celerity according to Stokes's linear theory. This provides an excellent dispersion accuracy up to $k'h' = 6$, i.e. twice the traditional deep water limit. The linear shoaling properties determined directly from the continuity equation and momentum equations are also shown to be in excellent agreement with fully dispersive linear theory up to $k'h' = 6$.

Next, the nonlinear properties are analysed focusing on second- and third-order transfer including amplitude dispersion for weakly nonlinear regular waves. In this context we assume ϵ to be small (and μ to be arbitrary) and use Stokes's third-order theory as the reference solution. We come to the remarkable conclusion that the dispersion-enhancement technique also improves the nonlinear transfer when a sufficient order of dispersion is retained also in the nonlinear terms. As an example, the expansion (from $k'h' = 0$) of Stokes's second-order target solution is matched with the enhanced $(\mu^4, \epsilon\mu^4)$ formulation to a degree, which would normally require that $(\mu^6, \epsilon\mu^4)$ terms were retained in the Boussinesq formulation. This result can neither be obtained if $\epsilon\mu^4$ terms are neglected nor if the enhancement is omitted. Similar conclusions apply to the lower-order Boussinesq equations.

In §§ 5 and 6 we shift to the alternative velocity variable, $\tilde{\boldsymbol{u}}$, taken at an arbitrary vertical location. First (in § 5), the original higher-order equations from § 2 are recast in terms of $\tilde{\boldsymbol{u}}$, assuming $\epsilon = O(1)$ and retaining terms of order $O(\mu^4, \epsilon^5\mu^4)$. The resulting equations contain fifth-derivative terms in the continuity equation as well as in the momentum equations. The arbitrary vertical location of $\tilde{\boldsymbol{u}}$ is used for optimizing the linear dispersion for $k'h' < 6$. Unfortunately, this single degree of freedom is not sufficient to match the Padé [4,4] expansion and the resulting dispersion characteristics are not as accurate as the ones obtained in § 4. The nonlinear transfer functions are, however, shown to be surprisingly accurate for $k'h' < 6$. The Padé [4,4] dispersion relation can be matched by further generalizing the choice of velocity variable (Schäffer & Madsen 1995*b*) but this has not been pursued in the present paper.

Next, we introduce the dispersion-enhancement technique from § 4. In this context, we leave out the $O(\mu^4)$ terms from § 5 and concentrate on a set of lower-order equations retaining $O(\mu^2, \epsilon^3\mu^2)$. The resulting enhanced equations are presented in § 6 and they contain third-derivative terms in the continuity equation as well as in the momentum equations. The analysis shows that these equations incorporate highly accurate linear dispersion and shoaling characteristics (of the same quality as the enhanced higher-order equations from § 4). The nonlinear transfer functions and the amplitude dispersion are also satisfactory although slightly less accurate than the results achieved in § 4. Again it is demonstrated that the dispersion-enhancement technique carries over to the nonlinear transfer when sufficient order of dispersion is retained in the nonlinear terms: the enhanced $O(\mu^2, \epsilon\mu^2)$ formulation is superior to the enhanced $O(\mu^2, \epsilon)$ formulation, which again is superior to the non-enhanced $O(\mu^2, \epsilon)$ formulation.

Section 7 contains a further analysis of the nonlinear properties of the various Boussinesq-type equations with focus on second-order sub- and superharmonics. Accepting a 10% error on the second-order transfer functions, the most advanced $\boldsymbol{U}$ equations including $O(\mu^4, \epsilon\mu^4)$ are shown to be applicable up to $h'/L'_{0,\mathrm{R}} = 0.53$, where $L'_{0,\mathrm{R}} = 2\pi g/\omega'^2_{\mathrm{R}}$ and ω_{R} is the geometrical mean of the two primary-wave angular frequencies involved. This is slightly over the traditional deep water limit. For equations in terms of $\boldsymbol{U}$ or $\tilde{\boldsymbol{u}}$ retaining $O(\mu^2, \epsilon\mu^2)$ this error limit is reached at $h'/L'_{0,\mathrm{R}}$ between 0.25 and 0.32. The 10% error limit is found already at $h'/L'_{0,\mathrm{R}}$ ranging from 0.06 to 0.09 when only $O(\mu^2, \epsilon)$ is retained. As expected, results are much improved by retaining higher-order dispersion in the nonlinear terms and again it is found that the dispersion-enhancement also betters the results for nonlinear transfer. Note that although higher-order nonlinearities are included in the equations, we have, in the present context, only mentioned the terms which are linear in ϵ, since terms of higher order in ϵ are not relevant for the second-order transfer.

Finally (in § 8), dispersion characteristics in connection with waves in ambient currents are analysed on the basis of the new Boussinesq equations in $\boldsymbol{U}$ and $\tilde{\boldsymbol{u}}$. It turns out that the highly accurate linear dispersion characteristics (Padé [4,4]) are retained in connection with the correct Doppler shift. In the classical Boussinesq equations where $\epsilon = O(\mu^2)$, the presence of an ambient current calls for special attention and scaling, as shown by Yoon & Liu (1989), who extended the equations of Peregrine (1967) to achieve the correct Doppler shift. In the present work, we have avoided this complication by taking ϵ as arbitrary and the equations will consequently incorporate the same linear dispersion characteristics for the relative wave motion with or without currents. The highly accurate dispersion obtained in connection with ambient currents represents a major step forward in comparison with the state-of-the-art, and it allows for a study of the phenomenon of wave blocking (provided that $F_c \leqslant -0.21$).

In conclusion, the derivations and analysis made in §§ 2–8 have resulted in the identification of two interesting sets of Boussinesq equations, both having excellent characteristics with regard to dispersion, shoaling and nonlinear properties. One set (§ 4) is expressed in terms of the depth-averaged velocity, $\boldsymbol{U}$, retaining $O(\mu^4, \epsilon\mu^4, \epsilon^3\mu^2)$, while the other set (§ 6) is given in terms of the particle velocity, $\tilde{\boldsymbol{u}}$, retaining $O(\mu^2, \epsilon^3\mu^2)$. The classical Boussinesq equations are limited by including only weak dispersion and nonlinearity. This typically limits accurate applications to a narrow region somewhat outside the surf zone. Inclusion of the new higher-order terms expands the application range significantly, covering the range from deep water all the way up to the surf zone. Further incorporation of wave breaking using the approach of Schäffer *et al*. (1993) and Madsen *et al*. (1997) is planned. This will provide a model applicable to the whole range from deep water to the shoreline including the surf and swash zones.

Due to their extensive length, the equations derived in this paper may not seem viable as basis for a numerical model. However, it is the complexity of the terms rather than the number of terms that matters when building a numerical model and during the review process of this paper, a numerical code solving these equations has in fact been developed and published by Madsen *et al*. (1996). While the computational cost is less than twice that of solving the classical Boussinesq equations, the improvement of the accuracy is immense, as predicted by the analyses in the present paper.

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Appendix A. Nomenclature

symbol	description	first appearance
$'$	primes denote dimensional quantities	(2.1)
a'_0	characteristic wave amplitude	(2.1)

a_1	first-order wave amplitude	(3.10)
a_2	second-order wave amplitude	(3.10)
a_3	third-order wave amplitude	(3.10)
a_p	second-order wave amplitude	(7.3 *b*)
$A = A(x)$	slowly varying wave amplitude	(4.12)
b_p	second-order wave amplitude	(7.3 *b*)
c_2, c_4	expansion coefficients	(3.23)
$D = D(x)$	slowly varying velocity amplitude	(4.12)
F_1, $F_1^{\pm}$	forcing function in mass equation	(3.18), (7.5)
F_2, $F_2^{\pm}$	forcing function in momentum equation	(3.18), (7.5)
F_c	Froude number	(8.14)
g	acceleration due to gravity	(2.1)
$G_{\eta}^{\pm}$	second-order transfer function for η	(7.3 *a*)
$G_{\boldsymbol{U}}^{\pm}$	second-order transfer function for $\boldsymbol{U}$	§ 7 *b*
h_0'	characteristic water depth	(2.1)
k	wavenumber	(3.10)
k_p	sum and difference wavenumber	(7.3 *c*)
l_0'	characteristic wavelength	(2.1)
$L_{0,\mathrm{R}}$		§ 7 *b*
$m_{ij}^{(1)}$	first-order coefficients, $i, j = 1, 2$	(3.13)
$m_{ij}^{(2)}$	second-order coefficients, $i, j = 1, 2$	(3.18)
$\boldsymbol{Q} = \boldsymbol{Q}(x, y, t)$	depth-integrated velocity, flux	(2.18)
t	time	(2.1)
$\boldsymbol{u} = \boldsymbol{u}(x, y, z, t)$	horizontal particle velocity	(5.1)
$\hat{\boldsymbol{u}} = \hat{\boldsymbol{u}}(x, y, t)$	velocity at the SWL	(2.9)
$\tilde{\boldsymbol{u}} \equiv \boldsymbol{u}(x, y, \tilde{z}, t)$	velocity at arbitrary z level	§ 5 *a*
$\tilde{u}_1$	first-order velocity amplitude	(5.6)
$\tilde{u}_2$	second-order velocity amplitude	(5.6)
$\tilde{u}_3$	third-order velocity amplitude	(5.6)
$\tilde{\boldsymbol{u}}^{\mathrm{C}}$	current part of $\tilde{\boldsymbol{u}}$	§ 8 *b*
$\tilde{\boldsymbol{u}}^{\mathrm{W}}$	wave part of $\tilde{\boldsymbol{u}}$	§ 8 *b*
$\boldsymbol{U} = \boldsymbol{U}(x, y, t)$	depth-averaged velocity	(3.1)
U_1	first-order velocity amplitude	(3.10)
U_2	second-order velocity amplitude	(3.10)
U_3	third-order velocity amplitude	(3.10)
$\boldsymbol{U}^{\mathrm{C}}$	current part of $\boldsymbol{U}$	§ 8 *a*
$\boldsymbol{U}^{\mathrm{W}}$	wave part of $\boldsymbol{U}$	§ 8 *a*
$w = w(x, y, z, t)$	vertical particle velocity	(2.8)
$\hat{w} = \hat{w}(x, y, t)$		(2.9)
$w^{(m)} = w^{(m)}(x, y, t)$		(2.12)
(x, y, z)	Cartesian coordinates	(2.1)
$\tilde{z} = \tilde{z}(x, y)$	z location at which $\tilde{\boldsymbol{u}}$ is taken	§ 5 *a*
$\alpha \equiv \dfrac{\tilde{z}}{h} + \dfrac{1}{2}\left(\dfrac{\tilde{z}}{h}\right)^2$		(5.7)
α_1, α_2	enhancement parameters	(4.1)
β_1, β_2	enhancement parameters	(4.1)

$\gamma, \gamma_n, \gamma_m$		(5.11), (7.9)
γ_0	shoaling gradient	(4.8)
$\gamma_1, \gamma_2, \gamma_3$	coefficients	(4.17)
γ_4, γ_5	coefficients	(4.20)
$\hat{\Gamma}$		(2.17)
Γ		(3.8 g)
$\tilde{\Gamma}$		(5.2 b)
$\bar{\Gamma}$		(5.5 k)
δ		(7.3 d)
ϵ	nonlinearity parameter	(2.4)
$\zeta \equiv \dfrac{\omega}{k\sqrt{h}} \left(= \dfrac{c'}{\sqrt{gh'}} \right)$	normalized celerity	(4.19)
η	free surface elevation	(2.1)
η_n, η_m	first-order free surface elevations	(7.1)
$\eta^{(2)}$	second-order surface elevation	(7.2)
$\eta_{nm}^{\pm}$		(7.3 a)
$\theta \equiv \omega_t - kx$		(3.10)
$\kappa \equiv \mu kh = k'h'$		(3.12)
$\Lambda_{nm}^{\mathrm{I}}$	Boussinesq equation terms of order $O(\mu^n \epsilon^m)$	(2.16)
$\Lambda_{nm}^{\mathrm{II}}$		(3.7)
$\Lambda_{nm}^{\mathrm{III}}$		(4.2)
$\Lambda_{nm}^{\mathrm{IV}}$		(5.4)
$\Lambda_{nm}^{\mathrm{V}}$		(6.3)
μ	dispersion parameter	(2.4)
$\sigma = \sigma(x)$	slowly varying phase shift between A and D	(4.12)
σ_1, σ_2	parameters	(5.10)
$\Phi = \Phi(x, y, z, t)$	velocity potential	(2.3)
$\Phi^{(n)} = \Phi^{(n)}(x, y, t)$		(2.5)
$\Psi = \Psi(x)$	phase function	(4.13)
ω	angular frequency	(3.10)
ω_p	sum and difference angular frequency	(7.3 c)
ω_{R}		§ 7 a
$\Omega_n \equiv \dfrac{\omega_n'}{2\pi}\sqrt{\dfrac{h'}{g}}$		§ 7 a
$\Omega_m \equiv \dfrac{\omega_m'}{2\pi}\sqrt{\dfrac{h'}{g}}$		§ 7 a

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